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ELEMENTS

OF

TRIGONOMETRY.

WITH

LOGARITHMIC AND OTHER TABLES.

BY

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WITH THE CO-OPERATION OF

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PREFACE.

In preparing this text-book, Colonel Edgar W. Bass, Professor of Mathematics in the U. S. Military Academy, West Point, N. Y., has been my collaborator, and the requirements of that Academy have almost entirely determined the extent and detail of treatment.

The trigonometric functions are more completely explained than usual, each being illustrated and discussed separately.

The usual chapter on the construction and use of trigonometric tables is omitted from the body of the work, and incorporated with the explanation of the tables.

The spherical triangle proper is carefully distinguished from its corresponding trihedral angle at the centre of the sphere.

Trigonometries by Chauvenet, Church, Olney, Newcomb, Wheeler, Todhunter, Beasley, and Hann have been freely consulted.

The method in Newcomb's Trigonometry for deducing the developments of sine x and cosine x have been closely followed.

Thanks are due to the instructors and students of mathematics in the U. S. Military Academy during the past three years, for many solutions and corrections.

The demonstration of the 7th principle of right trihedrals, p. 130, is by Lieutenant E. D. Smith, 19th U. S. Infantry, to whom I am also indebted for valuable suggestions.

Henry H. Ludlow.

New York City, April 15, 1890.

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GREEK ALPHABET.

Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	ο	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ϵ	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zeta	Σ	σ ς	Sigma
Η	η	Eta	Τ	τ	Tau
Θ	ϑ θ	Theta	ϒ	υ	Upsilon
Ι	ι	Iota	Φ	φ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

PART I.

PLANE TRIGONOMETRY.

CHAPTER I.

MEASUREMENT AND ESTIMATION OF ANGLES.

1. Every triangle is composed of six parts: three angles, and three sides.

Knowing certain parts, the remaining parts may be constructed and measured by methods explained in Geometry; but owing to instrumental and personal errors, such constructions and measurements are not always sufficiently accurate.

When great precision is desired, the unknown parts are computed by methods to be explained.

The computation of the unknown parts of a triangle from a sufficient number of known parts is called *the solution of the triangle.*

2. Trigonometry* is that branch of Mathematics which treats algebraically:

First. Of the measurement, and relations, of angles and their sides.

Second. Of the solution of triangles.

---

* The word Trigonometry is derived from two Greek words, one signifying *a triangle*, and the other *I measure*. Originally it was limited to the solution of triangles, but modern Trigonometry is extended as above indicated. More definitely speaking, Trigonometry is a subdivision of Geometry, but it is generally treated as a separate branch of Mathematics.

It is divided into two parts: **Plane Trigonometry**, which treats of plane angles and triangles; and **Spherical Trigonometry**, which treats of trihedrals and spherical triangles.

MEASUREMENT OF ANGLES.

3. An angle is measured by expressing it in terms of some angle assumed as the unit of measure.

In Geometry the right angle is generally taken as the angular unit.

4. The Sexagesimal Division, which is generally used in computations, consists of degrees, minutes, and seconds. A degree, written 1°, is $\frac{1}{90}$ of a right angle; a minute, written 1′, is $\frac{1}{60}$ of a degree; and a second, written 1″, is $\frac{1}{60}$ of a minute.

Exercises.

Express in degrees, minutes, and seconds, the interior angle, and the angle at the centre subtended by any side, in each of the following regular polygons:

	Answers.	
Triangle, . .	60°,	120°.
Square, . . .	90°,	90°.
Pentagon, . .	108°,	72°.
Hexagon, . .	120°,	60°.
Heptagon, .	128° 34′ 17″.143,	51° 25′ 42″.857.
Octagon, . .	135°,	45°.
Nonagon, .	140°,	40°.
Decagon, .	144°,	36°.
Undecagon,	147° 16′ 21″.82,	32° 43′ 38″.18.
Dodecagon,	150°,	30°.
Icosagon,	162°,	18°.

The *arc* of any circumference which subtends 1° is called a degree of arc, and is written, 1° of arc. It varies with the radius of the circle, and should not be confused with an angle of 1° which is constant. In the same or equal circles angles at the centre are proportional to the arcs which subtend them, and may therefore be compared by means of these arcs, but such is not the case, when circles have different radii.

The Centesimal Division, which was proposed in France, but not generally adopted, consists of grades, minutes, and seconds. A grade, written 1^g, is $\frac{1}{100}$ of a right angle; a minute, written $1^{`}$, is $\frac{1}{100}$ of a grade; and a second, written $1^{``}$, is $\frac{1}{100}$ of a minute.

$$90^\circ = 100^g, \quad \therefore \quad 1^\circ = \frac{10^g}{9}.$$

Therefore any number of degrees multiplied by $\frac{10}{9}$ will give the corresponding number of grades; and the product of any number of grades by $\frac{9}{10}$ will be the equivalent number of degrees.

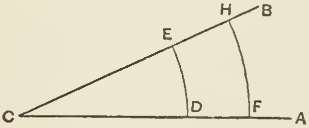
$$1' = \frac{50^{`}}{27}, \quad \text{and} \quad 1'' = \frac{250^{``}}{81}.$$

Exercises.

Express in grade measure	120°,	60°,	$30^\circ\ 40'\ 20''$,	$15^\circ\ 25'$.
" " degree "	$15^g\ 72^{`}\ 63^{``}$,		$93^g\ 85^{`}$,	$187^g\ 99^{`}\ 98^{``}$.
" " " "	$7^g\ 82^{`}\ 92^{``}$,		$47^g\ 0^{`}\ 17^{``}$,	$^{`}43^{`}\ 02^{``}$.
" " right angles	$59^g\ 92^{`}\ 87^{``}$,		$73^{`}\ 19^{``}$,	$75^{``}$.

5. The Radian Measure of an angle is the ratio of any circular arc subtending the angle to its radius.

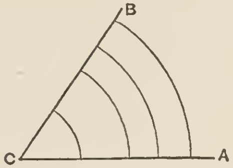
The radian measure of the angle ACB is $\dfrac{\text{arc } DE}{\text{radius } CD}$, or the equal ratio $\dfrac{FH}{CF}$.



An arc described with any convenient radius may be used, since the ratio of any arc subtending an angle to its radius does not vary with the radius. [Geometry.]

The radian measure, usually designated r-measure, contains the length of a circular arc, which can be exactly expressed only in terms of π. It is, therefore, frequently called the π-measure, or circular measure. Since π is an incommensurable number, approximately equal to 3.14159265, the r-measure is not suitable for practical computations or constructions; but it is generally used in discussions and deductions.

The unit of r-measure, called a radian, is the angle subtended by an arc equal in length to the radius with which it is described.



Angle ACB = a radian;
Arc = radius.

Since angle (in radians) $= \dfrac{\text{arc}}{\text{radius}}$, we have radius $= \dfrac{\text{arc}}{\text{angle}}$, and arc = radius × angle.

Exercises.

Find, in radians, the angles subtended by arcs of

1 ft., 5 yd., 3 fur., 1 mi., and 7 mi., respectively,

at 1 mi. (radius) from the vertex.

Ans. $\frac{1}{5280}$, $\frac{1}{352}$, $\frac{3}{8}$, 1, 7.

Find the radii with which arcs of 4 ft., 5 mi., and 3 yd. in length, respectively, subtend each of the following angles given in radians:

$$\alpha = \tfrac{3}{7}, \quad \beta = \tfrac{1}{9}, \quad \gamma = 0.15.$$

Ans.	For α the radii are	$9\frac{1}{3}$ ft.,	$11\frac{2}{3}$ mi.,	7 yd.
	For β " " "	36 ft.,	45 mi.,	27 yd.
	For γ " " "	$26\frac{2}{3}$ ft.,	$33\frac{1}{3}$ mi.,	20 yd.

Find the lengths, in feet of arcs, which at distances of 10 ft., 7 yd., and 4 mi., respectively, subtend each of the following angles:

$$\alpha = 1\tfrac{2}{3}, \quad \beta = \tfrac{1}{4}, \quad \gamma = .0035.$$

Ans.	For α the arcs are	$16\frac{2}{3}$ ft.,	35 ft.,	35200 ft.
	For β " " "	$2\frac{1}{2}$ ft.,	$5\frac{1}{4}$ ft.,	5280 ft.
	For γ " " "	.035 ft.,	.0735 ft.,	73.92 ft.

6. *To pass from degrees to radians or the reverse.* Since

$$180^\circ \text{ in radians} = \frac{\pi r}{r} = \pi = 3.14159265 \ldots ;$$

$$1^\circ = \frac{\pi}{180} = 0.017453293 \ldots, \text{ radians};$$

and $$x^\circ = x\left(\frac{\pi}{180}\right) \text{ radians.}$$

That is, *the number of radians in any angle, as ϕ, is equal to the number of degrees in ϕ, multiplied by $\frac{\pi}{180}$; and the number of degrees in ϕ is equal to the number of radians in ϕ, multiplied by $\frac{180}{\pi}$.*

Exercises in passing from degree to r-measure, and the reverse.

Show that $1'' = \pi \div 648000 = 0.000004848137$ etc.

$1' = \pi \div 10800 = 0.000290888209$ etc.

$1^\circ = 0.0174532925.$

$5^\circ = \pi \div 36 = 0.0872664626.$

$15^\circ = 0.2617993878.$ $6^\circ = 0.1047197551.$

$12^\circ = 0.2094395103.$ $18^\circ = 0.3141592654.$

$20^\circ = 0.3490658504.$

$57^\circ\ 17'\ 44''.8 = 1 = 57^\circ.2957795.$

$114^\circ.591559 = 2.$ $171^\circ.8873385 = 3.$

$90^\circ = \pi \div 2 = 1.57079632.$ $229^\circ.183118 = 4.$

$286^\circ.4788975 = 5.$ $343^\circ.774677 = 6.$

$270^\circ = 3\pi \div 2 = 4.71238897.$ $360^\circ = 6.28318531.$

$8^\circ\ 7' = 0.1416625.$ $5^\circ\ 37'\ 30'' = 0.09817527.$ $75^\circ = 1.30899694$

$-60^\circ = -(\pi \div 3) = -1.047197551.$

$-210^\circ = -(7\pi \div 6).$ $-300^\circ = -(5\pi \div 3).$

$540^\circ = 3\pi$. $720^\circ = 4\pi$. $3600^\circ = 20\pi = 62.83185307$.

$\frac{\pi}{6} = 30^\circ = 0.52359877$. $\frac{\pi}{5} = 36^\circ = 0.62831853$.

$0.6684031 = 38^\circ\ 17'\ 48''$.

$1 = 3437'.74677 = 206264''.806$.. $0.49372 = 28^\circ\ 17'\ 15''.6$.

$0.24685 = 14^\circ\ 8'\ 37''.8$. $0.78539816 = \frac{\pi}{4} = 45^\circ$. $10 = 572^\circ.957795$

$5\pi = 900^\circ$. $\frac{-3\pi}{4} = -135^\circ$.

$\frac{-(2\pi + 3)}{4} = -132^\circ.9718346$. $\frac{-5\pi}{3} = -300^\circ$.

Express in right angles

	$572^\circ\ 15'$,	$\frac{\pi}{3}$,	-243°,	$\frac{7\pi}{10}$.
Ans.	6.3582,	$\frac{2}{3}$,	-2.7,	1.4.

			Answers.
Express in radians	$\frac{2}{7}$	right angles . .	0.4488
	0.063	" " . .	0.09896
	-0.3	" " .	-0.471239
	-1.3	" " .	-2.042

Find the angles in degrees corresponding to the following radii and lengths of arcs:

Given.		*Required.*	*Given.*		*Required.*
Distance or Radius.	Arc.	Angle in Degrees.	Distance or Radius.	Arc.	Angle in Degrees.
2 mi.	80 ft.	$0^\circ.43406$	4 ft.	4 ft.	$57^\circ.2957795$
6 ft.	3 ft.	$28^\circ\ 38'\ 52''.4$	100 ft.	5 ft.	$2^\circ.86479$
500 ft.	25 ft.	$2^\circ\ 51'\ 53''.2$	1 yd.	3 yd.	$171^\circ.8873385$
1 yd.	1 ft.	$19^\circ\ 5'\ 54''.9$	1 yd.	1 in.	$1^\circ\ 35'\ 29''.6$
1 mi.	1 fur.	$7^\circ.16197$	1 mi.	1 yd.	$0^\circ.32554$
1 mi.	1 ft.	$0^\circ\ 0'\ 0''.4$	9 meters	4.2 centimeters.	

Find the arcs in feet subtending the following angles at the distances given:

Given.		*Required.*	*Given.*		*Required.*
Angle.	Distance.	Arc in Feet.	Angle.	Distance.	Arc in Feet.
60°	6 ft.	6.283	150°	2 yd.	15.71
330°	2 yd.	34.56	268°	2 yd.	18.71
135°	2 yd.	14.14	18°	2 yd.	1.885
1°	100 ft.	1.75	1′	100 ft.	0.029
1″	100 ft.	0.0048	0.1″	100000 ft.	0.484
6° 30′	100 ft.	11.34	1″	206264.806 ft.	10.0
1′	1 mi.	1.53589	1°	1 mi.	92.1534
$\frac{\pi}{7}$	$\sqrt{85}$ yd.	4.14	$\pi^{\frac{3}{5}}$	$50^{\frac{3}{5}}$ in.	1.73
1.3 radians	20 rds.	429.0	$\frac{2\pi - 3}{4}$	47 meters.	38.65 m.

Find the distance at which an arc of 32 feet subtends 32°

Since, distance (or radius) $= \dfrac{\text{arc}}{\text{angle in radians}}$,

and
$$32^\circ = 32 \times \frac{\pi}{180} \text{ radians;}$$

$$\therefore \text{ dis.} = \frac{32 \text{ ft.}}{32 \times \dfrac{\pi}{180}} = \frac{180}{\pi} \text{ ft.} = 57.2957795 \text{ ft.}$$

Find the distances (radii) in yards, at which the following angles will be subtended by the given arcs:

Given.		*Required.*	*Given.*		*Required.*
Angle.	Arc.	Distance or Radius in Yds.	Angle.	Arc.	Distance or Radius in Yds.
1°	1 ft.	19.09859	1′	1 yd.	3437.75
1″	1 in.	5729.579	1″	1 ft.	68754.935
6° 30′	100 ft.	293.825	1°	2 ft.	38.1966
30°	1 mi.	3359.84	270°	1760 yd.	373.5
$\frac{\pi}{10}$	5 ft.	5.3	$\frac{3\pi + 7}{19}$	½ mi.	1018.0
$\pi^{\frac{3}{2}}$	45 ft.	2.7	$\sqrt{\pi}$	5 in.	0 08
72 radians	1000 yds.	13.89	2.75 radians	45 ft.	5.45

1. The length of an arc subtending an angle is equal to that of its radius. Find the angle in seconds.

Ans. $206264''.806$, which is known as the number of seconds in radius.

2. The length of an arc is 63 ft., and its radius is 150 ft. Find the angles which two right lines, tangent, respectively, to the arc at its ends, make with each other.

Ans. $24^\circ.0639$ and $155^\circ.9361$.

3. Assume the earth's radius to be 3963 mi., and find the length, in miles, of 1° of arc on the equator, also $1'$ of arc. *Ans.* 69.167 mi.
1.153 mi.

4. If the angle, with vertex at the earth's centre, subtended by the sun's semi-diameter is $16'\ 2''$; and the distance from the earth's centre to that of the sun is 92885000 mi., what is the sun's diameter in miles approximately?* *Ans.* 866400 mi.

5. Find approximately the distance at which a man 6 ft. in height will subtend an angle of 1°. *Ans.* 343.77 ft.

6. If the angle, with vertex at the sun's centre, subtended by the earth's radius, assumed equal to 3963 mi., is $8''.8$, what is the distance from the earth's centre to that of the sun approximately? *Ans.* 92889000 mi.

7. The angle which a ship's course makes with the bearing of a light-house is 88°; 12 minutes later the bearing of the light makes an angle of 92° with the ship's course, which remains unchanged. If the ship runs 10 knots an hour, what is the distance in miles, approximately, from the ship to the light? † *Ans.* 33.0 mi.

8. If the angle, with vertex at the earth's centre, subtended by the moon's diameter is $31'.12745$; and the distance from the

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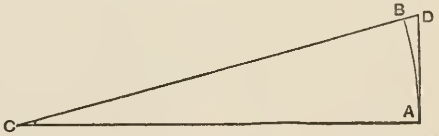
* The angle $ACB = \dfrac{\text{arc } AB}{CA}$.



At A draw AD perpendicular to CA. The error, due to taking $\dfrac{AD}{CA}$ as the measure of ACB,

will be less than $3''$ when $ACB = 2^\circ$, less than $\frac{1}{2}''$ when $ACB = 1^\circ$,
“ “ “ $\frac{1}{16}''$ “ “ $= 30'$, less than $\frac{1}{5000}''$ when $ACB = 1'$.

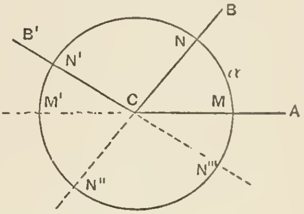
† A *knot* is a geographical mile, i.e., $1'$ of arc on the earth's equator.

earth's centre to that of the moon is 237780 mi., how will the earth's volume compare with that of the moon?

Ans. Earth's vol. = 49.892 moon's vol.
Diameter of moon = 2153 mi.

ESTIMATION AND SIGNS OF ANGLES.

7. Any plane angle, as ACB, may be considered as generated by the revolution of any convenient radius, as CM, about the vertex as a centre, from one side of the angle to the other. The side of the angle from which the radius is conceived to move is called the *initial* side, and the other the *final* side, of the angle. Thus, if CM rotates from CA to CB, CA is the initial, and CB the final side of ACB.



The order of the letters indicates the sides. Thus ACB indicates CA as the initial, and BCA indicates CB as the initial side.

An angle is estimated from its *initial* to its *final* side.

One complete revolution of the radius generates an angle of 360°. When the initial and final sides of an angle coincide, the angle is zero or some integral number of complete revolutions.

If two angles have a common initial side and differ by an integral number of complete revolutions, their final sides will coincide.

The outer end, M, of the radius describes a subtending arc, as MN, or NM, or MN', or MN'''.

The end of any subtending arc which is on the initial side is called its *origin*, and the end on the final side its *end*.

There are two directions of rotation, distinguished by the signs + and −. Either direction may be assumed as +, the other will then be −.

Unless it is otherwise stated, all angles estimated in the direction *opposite* to that of the hands of a clock will be considered as positive.

All distances, measured on the sides of angles in the direction from the vertex towards an end of the subtending arc, will be considered as positive, and those measured in an opposite direction, as negative.

Angles are estimated from the positive side of the initial to the positive side of the final side.

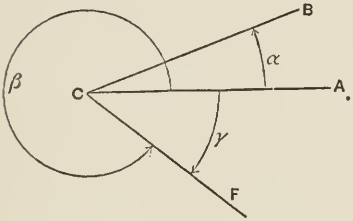
The angle ACB represents any angle positive or negative with CA as the initial and CB as the final side.

Representing the positive angle subtended by the arc MN by α, we have, with positive rotation,

$$ACB = \alpha, \quad \text{or} \quad \alpha + n360^\circ \ (n \text{ being any positive integer}):$$

and with negative rotation,

$$ACB = -(360^\circ - \alpha) = \alpha - 360^\circ, \quad \text{or} \quad \alpha - n360^\circ.$$



In order to avoid uncertainty, angles will be considered as less than 360°, unless it is otherwise indicated; and negative angles will be preceded by the sign —. Thus, ACB denotes the positive angle α, ACF the positive angle β, and $-ACF$ the negative angle γ.

It should be distinctly understood that algebraic signs as applied to angles *indicate direction of rotation only*, and do not affect values. 30° and − 30° are equal angles estimated in opposite directions.

Exercises.

With CA as the initial side and C as the vertex, construct right lines making the following angles:

$\pm 90^\circ$, $\pm 45^\circ$, $\pm 22^\circ\frac{1}{2}$, $\pm 30^\circ$, $\pm 15^\circ$, $\pm 7^\circ\frac{1}{2}$, $\pm 3^\circ\frac{3}{4}$

$\pm 810^\circ$, $\pm \frac{9}{4}\pi$, $\pm 382^\circ\frac{1}{2}$, $\pm \frac{13}{6}\pi$ $-\frac{23}{12}\pi$, $\pm 60^\circ$, $\pm 36^\circ$.

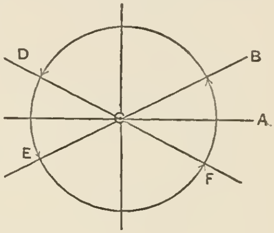
Make similar constructions with CB as the initial side.

Indicate in which direction positive distances should be laid off on each final side constructed.

Two arms attached to the same pivot, swing from the same position in opposite directions. The one moving in the positive direction, makes a complete revolution in 10 hrs., and the other in 3 hrs. What angle will each have generated when they first meet? *Ans.* $+83^\circ\frac{1}{13}$, and $-276^\circ\frac{12}{13}$.

When will they be together at the starting position, and what angle will each have generated?

8. An angle is said to be in the first, second, third, or fourth quadrant according as its final side lies in the first, second, third, or fourth of the right angles generated successively by the revolution of its initial side in a positive direction.



Thus—

$ACB < 90^\circ$ is in the 1st quadrant;
$ACD > 90^\circ$, and $< 180^\circ$ is in the 2d quad.;
$ACE > 180^\circ$, and $< 270^\circ$ " " 3d "
$ACF > 270^\circ$, and $< 360^\circ$ " " 4th "

The 5th quadrant is a repetition of the 1st, etc.

Exercises.

In what quadrants are the following angles?

$$192^\circ, \quad -45^\circ, \quad 1472^\circ, \quad -820^\circ, \quad 206264''.806,$$

$$\frac{\pi}{5}, \quad \frac{4\pi}{5}, \quad \frac{5\pi}{4}, \quad \frac{7\pi}{4}, \quad -\frac{\pi}{3}, \quad \frac{\pi+1}{2},$$

$$-\frac{2\pi}{3}, \quad -\frac{13\pi}{8}, \quad -3437'.74677.$$

9. **The Complement of an Angle** is the remainder obtained by subtracting it from 90°. Thus, the complement of 40° is 50°; that of 135° is −45°; and that of ϕ is $90^\circ - \phi$, or $\frac{\pi}{2} - \phi$, according as ϕ is in degree or r measure.

The Supplement of an Angle is the remainder obtained by subtracting it from 180°. Thus, the supplement of 60° is 120°; that of 200° is $-20°$; and that of ϕ is $180° - \phi$, or $\pi - \phi$, depending upon whether ϕ is in degree or r measure.

Exercises.

Angle.	Complement.
$30°$	$60°$
$375°$	$-285°$
$-30°$	$120°$
$\frac{\pi}{2}$	0
$\frac{\pi}{4}$	$\frac{\pi}{4}$
$\frac{\pi}{7}$	$\frac{5\pi}{14}$
7π	$\frac{-13\pi}{2}$
1.57079632	0
2	-0.42920368
$-\pi$	$\frac{3\pi}{2}$
4.71238897	$-\pi$
$20° + \phi$	$70° + \phi$
$135° - \phi$	$-45° + \phi$
$360° \pm \phi$	$-270° \mp \phi$
$7 - \pi$	$\frac{3\pi}{2} - 7$
-3.25	$4.82 +$

Angle.	Supplement.
$10°$	$170°$
$-5°$	$185°$
$210°$	$-30°$
$15° + \phi$	$165° - \phi$
$45° - \phi$	$135° + \phi$
$200° + \phi$	$-20° - \phi$
$375° - \phi$	$-195° + \phi$
$-\pi$	2π
$\frac{2\pi}{3} + 5$	$\frac{\pi}{3} - 5$
$\frac{7\pi}{4} - 2$	$-\frac{3\pi}{4} + 2$
$3.14159 + \pi$	0
-2.3	$5.34159 +$
$-\frac{2\pi - 3}{2}$	$\frac{4\pi - 3}{2}$

CHAPTER II.

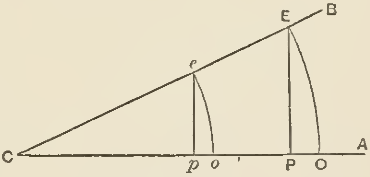
TRIGONOMETRIC FUNCTIONS OF AN ANGLE.

10. Any quantity whose value depends upon that of an angle is a function of the angle.

In Trigonometry eight functions of angles are employed, and as a class are called **Trigonometric Functions.**

They are named : **sine, cosine, tangent, cotangent, secant, cosecant, versed-sine,** and **coversed-sine;** and are abbreviated, respectively, to **sin, cos, tan, cot, sec, cosec, vers** or **versine,** and **covers** or **coversine.**

11. *The* **sine** *of an angle is the ratio to the radius, of the perpendicular distance of the end of the corresponding subtending arc from the initial side.*



Thus, let $ACB = a$ be any given angle. With any convenient radius, as $CO = R$, describe a subtending arc OE; and from the *end* of the arc draw EP perpendicular to the initial side CA. Then, $\sin a = \frac{PE}{R}$. Any other radius, as Co, with corresponding arc and perpendicular give the same result; since $\frac{pe}{Co} = \frac{PE}{R}$.

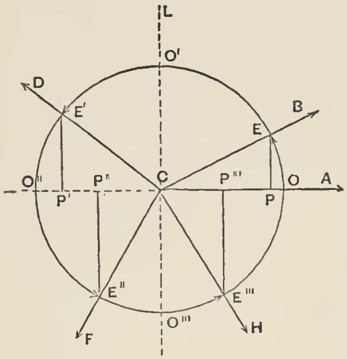
In some text-books the sine of an angle is defined to be the length of the above perpendicular expressed in terms of the corresponding radius as the unit of length; but PE expressed in terms of R is the ratio $\frac{PE}{R}$.

It should be observed that the sine of an angle is an abstract number, independent of the radius used; and dependent only upon the angle.

A radius and corresponding arc are used for convenience only. Some authorities make use of a right triangle, and define the sine of an angle to be the quotient of the opposite side by the hypothenuse

12. Algebraic Sign of the Sine of an Angle. Through the vertex of any given angle, as ACB, draw CL perpendicular to the initial side CA. With any convenient radius, as $CO = R$, describe the circum. $O\,O'O''O'''$. O is the origin or zero point of the arc ; O' is the 90° point, O'' is the 180° point, and O''' is the 270° point.

Distances estimated along perpendiculars to the initial side, in the direction of CO', that is from the vertex towards the 90° point, are considered as positive, and those in the opposite direction as negative. PE and $P'E'$ are positive, $P'''E'''$ and $P''E''$ are negative.



The radius is always considered as positive.

$$\text{Hence,}\quad \sin ACB = \frac{PE}{R}, \qquad \sin ACD = \frac{P'E'}{R},$$

$$\sin ACF = -\frac{P''E''}{R}, \quad \sin ACH = -\frac{P'''E'''}{R}$$

The sines of all positive angles in the 1st and 2d quadrants are positive, and of those in the 3d and 4th quadrants are negative.

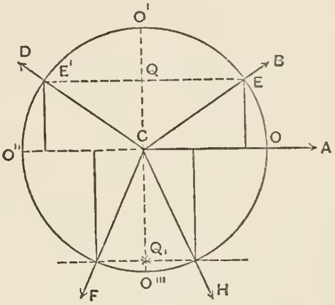
13. The sine of any given angle may be determined by construction and measurement, as follows: *With any convenient radius describe an arc subtending the angle, and from the end of the arc draw a perpendicular to the initial side. The quotient of the length of this perpendicular, with its proper sign, by that of the radius will be the required sine.*

For convenience and accuracy, a radius should be assumed which can be measured exactly by the scale used.

The sine of an angle being known, to construct a corresponding angle.

Assume the vertex and initial side. With any convenient radius, and the vertex as a centre, describe a circle. Through the vertex draw a right line perpendicular to the initial side. Lay off from the vertex upon this perpendicular, in that direction which corresponds with the sign of the given sine, a distance equal to the product of the radius by the given sine. Through the point thus determined draw a right line parallel to the initial side. Through the vertex and each of the two points in which this parallel intersects the circumference, draw a right line. Either will be the final side of a required angle.

Thus, having $\sin \phi = \frac{1}{2}$, to construct ϕ. Assume C as the vertex, and CA as the initial side. With a radius $CO = .6$ in. describe the circum. $OO'O''$. Draw $O'''CO'$ perpendicular to CA. Lay off $CQ = \frac{1}{2}R = .3$ in. Draw QE parallel to CA. Through C draw the right lines CE and CE'. ϕ will be either ACB or ACD.



In a similar manner, having $\sin \theta = -\frac{5}{6}$, we construct θ by laying off $CQ_, = -\frac{5}{6}R = -.5$ in., and find θ to be either of the angles ACF or ACH.

Between 0° and 360°, there are always two, and only two positive angles which correspond to each possible sine.

The sum of the two arcs of the same circumference which subtend supplementary angles having the same initial side and vertex, is a semi-circumference. It follows that the ends of such arcs are always in a right line parallel to the initial side.

Hence. *the sines of supplementary angles are equal.*

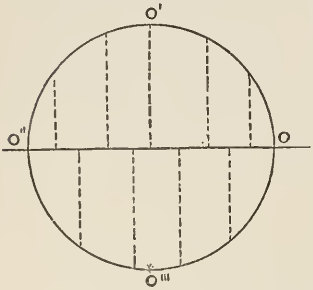
Representing any angle by a, we have

$$\sin a = \sin (180^\circ - a), \qquad \sin (90^\circ + a) = \sin (90^\circ - a),$$
$$\sin (180^\circ + a) = \sin (-a), \qquad \sin (270^\circ - a) = \sin (-90^\circ + a).$$

If two angles have equal sines it does not follow that they are supplementary unless each is less than 180°, for ACF and ACH have equal sines and are not supplementary.

Designating any angle as FCO''', which is equal to $O'''CH$, by β, we have $\sin(270° - \beta) = \sin(270° + \beta)$.

14. Variation of the Sine, as the Angle Changes. Conceive the final side to revolve from coincidence with the initial side, about the vertex as a centre, 360° in a positive direction. The angle will increase from 0° to 360°.



Since any assumed radius remains constant, the sine varies directly with the length of the perpendicular in the numerator of its measure.

As the angle increases from 0° to 90°, the perpendicular increases from 0 to R, hence the sine increases from 0 to 1.

From 90° to 180°, the perpendicular decreases from R to 0, hence the sine decreases from 1 towards 0.

From 180° to 270°, the perpendicular changes its sign from plus to minus and decreases from 0 to $-R$, hence the sine decreases from 0 to -1.

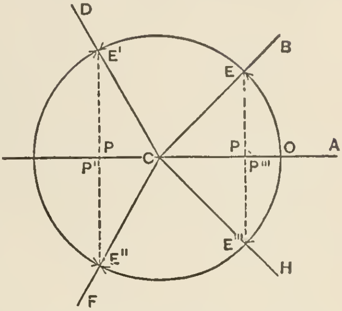
From 270° to 360°, the perpendicular increases from $-R$ to 0, hence the sine increases from -1 towards 0.

15. If the revolution be continued beyond 360°, the end of the corresponding subtending arc will coincide with that of some angle with the same origin less than 360°, and have the same sine.

Designating the latter angle by a, and any integer by n, we have $\sin(n360° + a) = \sin a$. Hence if an angle is greater than 360°, subtract 360° as many times as possible and take the sine of the remainder; thus, $\sin 1000° = \sin(1000° - 720°) = \sin 280°$

Therefore, if an angle continually increases, the sines will repeat themselves—varying from 1, the greatest possible value, through zero, to -1, the least possible value. Such a function is said to be *periodic*.

It should be observed that the perpendicular, and therefore the sine, do not, in general, change by equal amounts for equal changes in the angle. The sine changes most rapidly when the angle is passing through 0° or 180°, and least rapidly when passing through 90° or 270°



16. *Sines of Negative Angles.* Consider any negative angle with an equal positive angle having the same initial side, as $-ACH$ and ACB, or $-ACF$ and ACD, or $-ACD$ and ACF, etc.

It is apparent that in each pair, the corresponding perpendiculars, $P'''E'''$ and PE, $P''E''$ and $P'E'$, $P'E'$ and $P''E''$, etc., are equal with contrary signs. Hence, designating any angle by ϕ, we have $\sin(-\phi) = -\sin\phi$.

Exercises.

By construction and measurement determine the sine of each of the following angles:

$45°$, $30°$, $-90°$, $\frac{2}{3}\pi$, $15°$, $60°$, $-135°$,
$105°$, $\frac{5}{6}\pi$, $-180°$, $36°$, $18°$, $-150°$, $315°$,
$-120°$, $225°$, $210°$, $240°$, $-300°$, $330°$, $405°$,
$-810°$, $-1140°$, $-390°$, $900°$, $1200°$, $10\frac{1}{4}\pi$.

Which is the greater, $\sin 45°$ or $\frac{1}{2}\sin 90°$?
" " " $\sin 30°$ or $\frac{1}{2}\sin 60°$?
" " " $\sin 45°$ or $\sin 135°$?
" " " $\sin 230°$ or $\sin(-50°)$?
" " " $\sin 260°$ or $\sin(-80°)$?
" " " $\sin 220°$ or $\sin 320°$?
" " " $\sin 170°$ or $\sin 370°$?

What positive angles less than 360° have the same sines as the following?

$50°$, $140°$, $195°$, $275°$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{7\pi}{6}$.

As the angle increases, which is changing the more rapidly ?

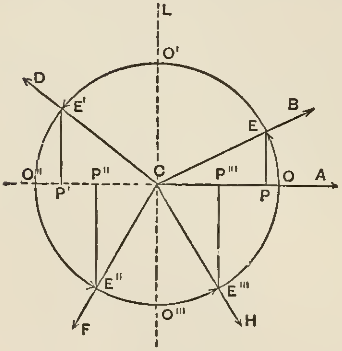
sin 30°	or	sin 40° ?	sin 45°	or	sin 135° ?
sin 60°	or	sin 150° ?	sin 160°	or	sin 220° ?
sin 70°	or	sin 290° ?	sin 190°	or	sin 350° ?
sin 185°	or	sin (−5°) ?	sin 130°	or	sin 230° ?

Construct both positive angles less than 360° which correspond to each of the following sines :

$$\tfrac{1}{3},\quad \tfrac{5}{7},\quad \sqrt{\tfrac{1}{2}},\quad -\tfrac{3}{4},\quad \tfrac{2}{3},\quad -.4,\quad -.27,\quad \tfrac{7}{8},$$

$$\frac{\sqrt{5}-1}{4},\quad \tfrac{1}{2}\sqrt{3},\quad -\sqrt{\tfrac{1}{2}},\quad -\tfrac{1}{2}\sqrt{3}.$$

17. *The* **cosine** *of an angle is the ratio to the radius, of the distance from the vertex to the foot of the perpendicular through the end of the corresponding subtending arc, to the initial side.*



Thus, with radius $CO = R$,

$$\cos ACB = \frac{CP}{R},$$

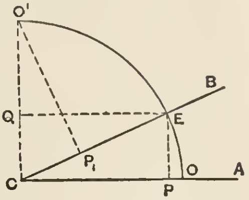
$$\cos ACD = -\frac{CP'}{R},$$

$$\cos ACF = -\frac{CP''}{R},$$

$$\cos ACH = \frac{CP'''}{R}.$$

Since distances estimated on an initial side, in the direction from the vertex towards the origin of any subtending arc are considered as positive, and those in the opposite direction as negative, the cosines of all positive angles in the 1st and 4th quadrants are positive, and of those in the 2d and 3d quadrants are negative.

18. Let ACB be a given angle. Through its vertex draw CO' perpendicular to CA. With C as a centre, and any convenient radius, as $CO = R$, describe the arc OEO'. Draw EP, EQ and $O'P_{\prime}$, respectively, perpendicular to CA, CO' and CB. The triangles $CP_{\prime}O'$ and CQE are equal in all their parts; hence



$$P_{\prime}O' = QE = CP$$

and

$$\cos ACB = \frac{CP}{R} = \frac{P_{\prime}O'}{R} = \sin BCO'.$$

BCO' is the complement of ACB; and since a similar construction with *any* given angle, as ϕ, will give a like result,* we have $\cos \phi = \sin (90° - \phi)$. That is—

The cosine of any angle is equal to the sine of its complement.

The sine and cosine are called complementary functions.

19. The cosine of any given angle may be determined by making a construction similar to that indicated for the sine. *The quotient of the distance, with its proper sign, from the vertex to the foot of the perpendicular, by the length of the radius will be the required cosine.*

The cosine of an angle being given, to construct a corresponding angle. *Assume the vertex and initial side, and describe a convenient circle. From the vertex lay off a distance, in the proper direction upon the initial side, equal to the product of the radius by the given cosine. Through the point thus determined draw a perpendicular to the initial side. Through the vertex and each of the two points in which this perpendicular intersects the circumference draw a right line. Either will be the final side of a required angle.*

Between 0° and 360° there are always two positive angles which correspond to each possible cosine.

Since the ends of the subtending arcs of the two angles are always in the same perpendicular to the initial side, it follows that the sum of the angles is always 360°. Hence, denoting any angle by ϕ, we have $\cos \phi = \cos (360^\circ - \phi)$.

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* Students should make the construction for an angle in each quadrant.

20. Variation of the Cosine as the Angle Changes. The cosine varies directly with the distance from the vertex to the foot of the perpendicular above described.

As the angle increases from 0° to 90°, this distance decreases from R to 0; hence the cosine decreases from 1 towards 0.

From 90° to 180°, the distance changes its sign from plus to minus, and decreases from 0 to $-R$; hence the cosine decreases from 0 to -1.

From 180° to 270°, the distance increases from $-R$ to 0; hence the cosine increases from -1 towards 0.

From 270° to 360°, the distance changes its sign from minus to plus, and increases from 0 to R; hence the cosine increases from 0 to 1.

If the angle is greater than 360°, we have

$$\cos (n360^\circ + a) = \cos a,$$

in which n is any integer, and a an angle less than 360°.

The cosine varies from 1 to -1, most rapidly when the angle is passing through 90° or 270°, and least rapidly when passing through 0° or 180°.

21. *Cosines of Negative Angles.* With a common initial side, equal angles with contrary signs have the ends of corresponding subtending arcs in the same perpendicular to the initial side. The distances from the vertex to the foot of each will coincide. Hence, denoting any angle by ϕ, we have

$$\cos (-\phi) = \cos \phi.$$

Exercises.

By construction and measurement determine the cosine of each of the following angles:

30°, 45°, 60°, -120°, 135°, -150°, $\frac{3}{4}\pi$,
$\frac{5}{4}\pi$, $-\frac{2}{3}\pi$, 225°, 300°, -780°, 1140°.

Show that $\cos 45^\circ > \frac{1}{2}\cos 0^\circ$, $\cos 30^\circ > \frac{2}{3}\cos 0^\circ$,
$\cos 60^\circ > \frac{1}{3}\cos 0^\circ$, $\cos 60^\circ > \frac{1}{2}\cos 30^\circ$.

What positive angles less than 360° have the same cosines as the following?

18°, $\frac{2}{3}\pi$, 115°, 210°, 150°, $\frac{5}{4}\pi$, 290°, -20°, -300°.

As the angle varies, which changes the more rapidly?

$\cos 10^\circ$ or $\cos 80^\circ$? $\cos 60^\circ$ or $\cos 210^\circ$?
$\cos 100^\circ$ or $\cos 290^\circ$? $\cos -70^\circ$ or $\cos 110^\circ$?
$\cos -300^\circ$ or $\cos 130^\circ$? $\cos -50^\circ$ or $\cos -140^\circ$?

Construct both positive angles less than 360° which correspond to each of the following cosines:

$\frac{1}{3}$, $\frac{1}{2}$, $.7$, $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3}}{2}$, $-\frac{2}{5}$, $-\frac{1}{\sqrt{2}}$, $-\frac{\sqrt{3}}{2}$, $.3$, $-\frac{4}{5}$.

22. *The* **tangent** *of an angle is the ratio to the radius, of the distance, measured on a tangent line to the corresponding circumference at its origin, from the origin to the final side.*

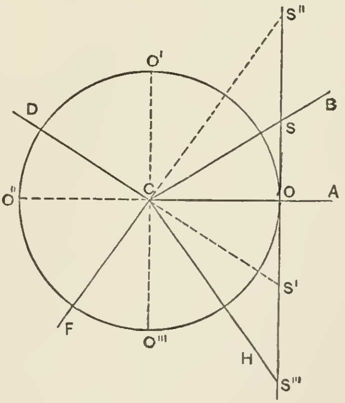
Thus, with radius $CO = R$,

$$\tan ACB = \frac{OS}{R},$$

$$\tan ACD = -\frac{OS'}{R};$$

$$\tan ACF = \frac{OS''}{R},$$

$$\tan ACH = -\frac{OS'''}{R}.$$



The tangents of all positive angles in the 1st and 3d quadrants are positive, and in the 2d and 4th, negative.

23. The tangent of any given angle may be determined as follows: *At the origin of any subtending arc draw a tangent line. The quotient of the distance with its proper sign, from the origin to the point where this tangent line intersects the final side, by the length of the radius will be the required tangent.*

The tangent of an angle being known, to construct a corresponding angle.

Assume the vertex, initial side, and describe as before a convenient circle. At the origin of the arc draw a tangent line to it. Lay off upon it, from the origin, a distance equal to the product of the radius by the given tangent: the direction depending upon the sign of the tangent. Through the point thus determined, and the vertex, draw a right line; it will be the final side of a required angle.

The portion of this line beyond the vertex will also be a final side of another angle having the same tangent.

Between 0° and 360° there are always two positive angles which correspond to each tangent.

Since the ends of the subtending arcs of these two angles are always in the same right line through the vertex, it follows that they differ by 180°. Hence, denoting any angle by ϕ, we have

$$\tan\phi = \tan(180^\circ + \phi).$$

24. Variation of the Tangent as the Angle Changes. The tangent varies directly with the distance from the origin of the subtending arc, to the point where the final side intersects the tangent line to the arc at the origin.

As the angle increases from 0° to 90°, this distance increases from 0 to $+\infty R$; hence the tangent increases from 0 towards $+\infty$.

From 90° to 180°, the distance changes its sign from plus to minus, and increases from $-\infty R$ to 0; hence the tangent increases from $-\infty$ towards 0.

From 180° to 270°, the distance changes its sign from minus to plus, and increases from 0 to $+\infty R$; hence the tangent increases from 0 towards $+\infty$.

From 270° to 360°, the distance changes its sign from plus to minus, and increases from $-\infty R$ to 0; hence the tangent increases from $-\infty$ towards 0.

If the angle is greater than 360°, we have

$$\tan(n360^\circ + a) = \tan a.$$

The tangent varies from ∞ to $-\infty$, most rapidly when the angle is passing through 90° or 270°, and least rapidly when passing through 0° or 180°.

25. *Tangents of Negative Angles.* With the same initial side, the final sides of equal angles with opposite signs intersect the tangent at the origin of any subtending arc at equal distances from the origin, but on opposite sides. Hence, denoting any angle by ϕ, we have

$$\tan(-\phi) = -\tan\phi.$$

Exercises.

By construction and measurement determine the tangent of each of the following angles:

$30^\circ,\quad 45^\circ,\quad -60^\circ,\quad 120^\circ,\quad 135^\circ,\quad -150^\circ,$

$\frac{3\pi}{4},\quad \frac{\pi}{5},\quad -\frac{\pi}{12},\quad \frac{\pi}{10},\quad \frac{5\pi}{4},\quad -\frac{4\pi}{6},$

$225^\circ,\quad 330^\circ,\quad -315^\circ,\quad -765^\circ,\quad 1140^\circ.$

Show that $2\tan 30^\circ < \tan 60^\circ$.

What positive angles less than 360° have the same tangents as the following?

$10^\circ,\quad 50^\circ,\quad 113^\circ,\quad -25^\circ,\quad 290^\circ,\quad -300^\circ,\quad -336^\circ.$

As the angle varies, which is changing the more rapidly?

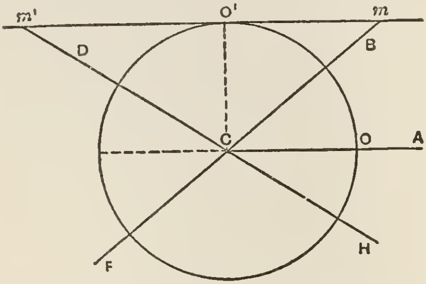
$\tan 27^\circ$ or $\tan 28^\circ$? $\tan 60^\circ$ or $\tan 120^\circ$? $\tan 75^\circ$ or $\tan 110^\circ$?
$\tan 170^\circ$ or $\tan -10^\circ$? $\tan 55^\circ$ or $\tan 230^\circ$? $\tan -200^\circ$ or $\tan 15^\circ$?

Construct both positive angles less than 360° which correspond to each of the following tangents:

$\frac{1}{2},\ -\frac{2}{3},\ .4,\ 1,\ \frac{1}{\sqrt{3}},\ \sqrt{3},\ 1.5,\ 2,\ 3\frac{1}{3},\ -\sqrt{3},\ -1,\ -\frac{1}{\sqrt{3}},\ -3.$

26. *The* cotangent *of an angle is the ratio to the radius, of the distance, measured on a tangent line to the corresponding circumference, at the "90° point," from that point to the final side.*

Thus, with radius $CO = R$,



$$\cot ACB = \frac{O'm}{R},$$

$$\cot ACD = -\frac{O'm'}{R},$$

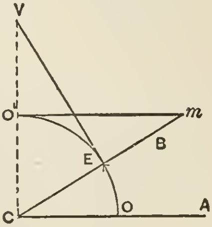
$$\cot ACF = \frac{O'm}{R},$$

$$\cot ACH = -\frac{O'm'}{R}.$$

The cotangents of all positive angles in the 1st and 3d quadrants are positive, and of all in the 2d and 4th are negative.

27. Let ACB be a given angle, and O' a "90° point."

At E and O' draw the tangents EV and $O'm$ respectively.



The triangles $CO'm$ and CEV are equal in all their parts· hence

$$EV = O'm, \quad \text{and} \quad \cot ACB = \tan BCV.$$

BCV is the complement of ACB; and since a similar construction with any given angle, as ϕ, will give a like result, we have

$$\cot \phi = \tan (90^\circ - \phi). \quad \text{That is—}$$

The cotangent of any angle is equal to the tangent of its complement.

The tangent and cotangent are complementary functions.

28. The cotangent of any given angle may be determined by making a construction similar to that indicated for the tangent, except that the 90° point must be used instead of the origin.

An angle corresponding to a given cotangent may be constructed in a manner similar to that described for a given tangent, substituting the 90° point for the origin, and the cotangent for the tangent.

Between 0° and 360° there are always two positive angles which correspond to each cotangent; and since the ends of the subtending arcs of these angles are always in the same right line through the vertex, it follows that they differ by 180°. Hence, denoting any angle by ϕ, we have $\cot\phi = \cot(180° + \phi)$.

29. Variation of the Cotangent as the Angle Changes. The cotangent varies directly with the distance from the 90° point to the point of intersection of the final side with the tangent line at the 90° point. As the angle increases from 0° to 90°, this distance decreases from ∞R to 0; hence the cotangent decreases from ∞ towards 0.

From 90° to 180°, the distance changes its sign from plus to minus, and decreases from 0 to $-\infty R$; hence the cotangent decreases from 0 towards $-\infty$.

From 180° to 270°, the distance changes its sign from minus to plus, and decreases from ∞R to 0; hence the cotangent decreases from ∞ towards 0.

From 270° to 360° the distance changes its sign from plus to minus, and decreases from 0 to $-\infty R$; hence the cotangent decreases from 0 towards $-\infty$.

If the angle is greater than 360°, we have $\cot(n360° + a) = \cot a$.

The cotangent varies from ∞ to $-\infty$, most rapidly when the angle is passing through 0° or 180°, and least rapidly when passing through 90° or 270°.

30. *Cotangents of Negative Angles.* With a common initial side, the final sides, of equal angles with contrary signs, intersect the tangent at the 90° point, at equal distances from that point, but on opposite sides. Hence, denoting any angle by ϕ, we have

$$\cot(-\phi) = -\cot\phi.$$

Exercises.

By construction and measurement determine the cotangent of each of the following angles:

$30°,\quad 45°,\quad -60°,\quad \tfrac{3}{4}\pi,\quad 120°,\quad 225°,\quad -330°,\quad 780°.$

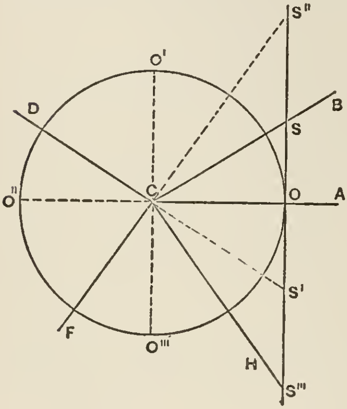
What positive angles less than 360° have the same cotangents as the following?

$$25^\circ,\quad \frac{\pi}{4},\quad 120^\circ,\quad \frac{2}{3}\pi,\quad 240^\circ,\quad \frac{3}{2}\pi,\quad -300^\circ,\quad -18^\circ.$$

Construct both positive angles, less than 360°, which correspond to each of the following cotangents:

$$3,\quad \sqrt{3},\quad -1,\quad \frac{1}{2}.\quad \frac{1}{\sqrt{3}},\quad -\frac{2}{3},\quad -.7\quad -\frac{1}{\sqrt{3}},\quad -\sqrt{3},\quad 2.$$

31. *The* **secant** *of an angle is the ratio to the radius, of the distance, measured on the final side, from the vertex to the tangent line to the corresponding subtending arc at its origin.*



Thus, with radius $CO = R$,

$$\sec ACB = \frac{CS}{R},$$

$$\sec ACD = -\frac{CS'}{R},$$

$$\sec ACF = -\frac{CS''}{R},$$

$$\sec ACH = \frac{CS'''}{R}.$$

Since distances on the final side of an angle are positive when measured in the direction from the vertex towards the end of the subtending arc, and negative in the opposite direction; the secants of all positive angles in the 1st and 4th quadrants are positive, and of all in the 2d and 3d quadrants are negative.

32. The secant of any given angle may be determined as follows: *At the origin of any subtending arc draw a tangent line to it. The quotient of the distance, with its proper sign, from the vertex to the intersection of the final side with the tangent line by the length of the radius, will be the required secant.*

The secant of an angle being known, to construct a corresponding angle.

Assume the vertex and initial side, and describe a convenient circle.

At the origin of the arc draw a tangent line. Multiply the given secant by the radius, and with the product as a new radius, and the vertex as a centre, describe an arc cutting the tangent line, in general, in two points. Through each of these two points and the vertex draw a right line. Four positive angles, one in each quadrant, will thus be formed having the assumed initial side. The angle in the 1st *quadrant or that in the* 4th *quadrant will be the one required if the given secant is positive; otherwise the one in the* 2d *and the one in the* 3d *will correspond to the given secant.*

Two angles, each less than 360°, correspond to each possible secant.

Since the ends of the subtending arcs of these two angles are always in the same perpendicular to the initial side, it follows that the sum of the angles is always 360°. Hence, denoting any angle by ϕ, we have

$$\sec \phi = \sec (360^\circ - \phi).$$

33. Variation of the Secant as the Angle Changes. The secant varies directly with the distance, on the final side, from the vertex to the tangent line at the origin. As the angle increases from 0° to 90°, this distance increases from R to ∞R; hence the secant increases from 1 to $+\infty$.

From 90° to 180°, the distance changes its sign from plus to minus, and increases from $-\infty R$ to $-R$; hence the secant increases from $-\infty$ to -1.

From 180° to 270°, the distance decreases from $-R$ to $-\infty R$; hence the secant decreases from -1 towards $-\infty$.

From 270° to 360°, the distance changes its sign from minus to plus, and decreases from ∞R to R; hence the secant decreases from ∞ to 1.

If the angle is greater than 360°, we have $\sec (n360^\circ + a) = \sec a$.

The secant varies from 1 to ∞, and from -1 to $-\infty$. It has no value between 1 and -1.

It varies most rapidly when the angle is passing through 90° and 270°, and least rapidly when passing through 0° and 180°.

34. *Secants of Negative Angles.* With the same initial side, the final sides, of equal angles with contrary signs, intersect the tangent at the origin of any subtending arcs at equal distances from the *vertex.* These distances are both positive when the angles are between 0° and ± 90°, and both negative when the angles are between ± 90° and ± 270°. Hence

$$\sec(-\phi) = \sec\phi.$$

Exercises.

By construction and measurement determine the secant of each of the following angles :

$30°$, $45°$, $-60°$, $120°$, $135°$, $-150°$, $\frac{3}{4}\pi$, $\frac{5}{4}\pi$,

$\frac{-4\pi}{6}$, $225°$, $-300°$, $-780°$, $1225°$.

Show that $\sec 140° = \sec 220°$, $\sec 60° = \sec 300°$.

What positive angles less than 360° have the same secants as the following ?

$15°$, $105°$, $260°$, $335°$, $-40°$, $-190°$, $-325°$.

As the angle varies, which is changing the more rapidly ?

$\sec 25°$ or $\sec 30°$? $\sec -140°$ or $\sec 150°$? $\sec 100°$ or $\sec -300°$?

Construct both positive angles less than 360° which correspond to each of the following secants :

$\frac{2}{\sqrt{3}}$, $\sqrt{2}$, 2, -1, $-\sqrt{2}$, $-\frac{2}{\sqrt{3}}$, 1.5, 3, -2.2.

35. *The* **cosecant** *of an angle is the ratio to the radius, of the distance measured on the final side from the vertex to the tangent line to the corresponding circumference at its 90° point.*

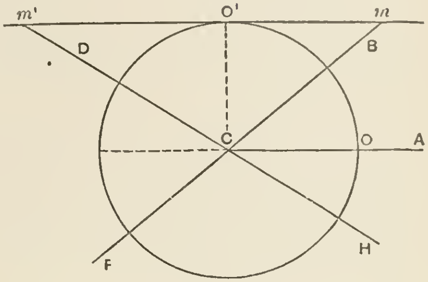
Thus, with radius $CO = R$,

$$\operatorname{cosec} ACB = \frac{Cm}{R},$$

$$\operatorname{cosec} ACD = \frac{Cm'}{R},$$

$$\operatorname{cosec} ACF = -\frac{Cm}{R},$$

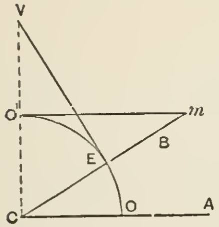
$$\operatorname{cosec} ACH = -\frac{Cm'}{R}.$$



The cosecants of all positive angles in the 1st and 2d quadrants are positive, and of all in the 3d and 4th are negative.

36. Let ACB be a given angle, and O' a 90° point. At E and O' draw the tangents EV and Om respectively. The triangles $CO'm$ and CEV are equal in all their parts; hence $Cm = CV$, and cosec ACB = sec BCV. BCV is the complement of ACB; and since a similar construction with any angle, as ϕ, will give a like result, we have

$$\operatorname{cosec} \phi = \sec (90° - \phi). \text{ That is—}$$



The cosecant of any angle is equal to the secant of its complement.

The secant and cosecant are complementary functions.

37. The cosecant of any given angle may be determined by making a construction similar to that indicated for the secant, except that the 90° point must be used instead of its origin.

An angle corresponding to a given cosecant may be constructed in a manner similar to that described for a given secant, substituting the 90° point for the origin, and the cosecant for the secant.

Between 0° and 360° there are always two positive angles which correspond to each possible cosecant. One is in the first quadrant, and the other is its supplement if the given cosecant is positive. One is in the third and the other in the fourth quadrant, but not supplementary, if the cosecant is negative.

38. Variation of the Cosecant as the Angle Changes. The cosecant varies directly with the distance from the vertex to the intersection of the final side with the tangent line at the 90° point.

As the angle increases from 0° to 90°, this distance decreases from ∞R to R; hence the cosecant decreases from ∞ to 1.

From 90° to 180°, the distance increases from R to ∞R; hence the cosecant increases from 1 towards ∞.

From 180° to 270°, the distance changes its sign from plus to minus, and increases from $-\infty R$ to $-R$; hence the cosecant increases from $-\infty$ to -1.

From 270° to 360° the distance decreases from $-R$ to $-\infty R$, hence the cosecant decreases from -1 towards $-\infty$.

If the angle is greater than 360°, we have

$$\operatorname{cosec}(n360^\circ + a) = \operatorname{cosec} a.$$

The cosecant varies from ∞ to 1, and from -1 to $-\infty$, and cannot have a value between 1 and -1.

It varies most rapidly when the angle is passing through 0° and 180°, and least rapidly when passing through 90° and 270°.

39. *Cosecants of Negative Angles.* With a common initial side, the final sides of equal angles with contrary signs intersect the tangent at the 90° point at equal distances from the vertex, but these distances always have contrary signs. Hence, denoting any angle by ϕ, we have

$$\operatorname{cosec}(-\phi) = -\operatorname{cosec}\phi.$$

Exercises.

By construction and measurement determine the cosecant of each of the following angles :

30°, -45°, 60°, $\frac{3}{4}\pi$, 120°, 240°, 300°, 540°.

What positive angles, less than 360°, have the same cosecants as the following?

$$10°, \quad -75°, \quad \frac{7\pi}{12}, \quad 135°, \quad -\frac{\pi}{6}, \quad \frac{\pi}{4}, \quad 405°.$$

Construct both positive angles, less than 360°, which correspond to each of the following cosecants:

$$2, \quad \sqrt{2}, \quad \frac{2}{\sqrt{3}}, \quad 3, \quad -1.5, \quad -2, \quad -\frac{2}{\sqrt{3}}, \quad -\sqrt{2}.$$

40. *The* **versed-sine** *of an angle is the ratio to the radius, of the distance of the origin of the corresponding subtending arc from the foot of the perpendicular, through the end of the arc, to the initial side.*

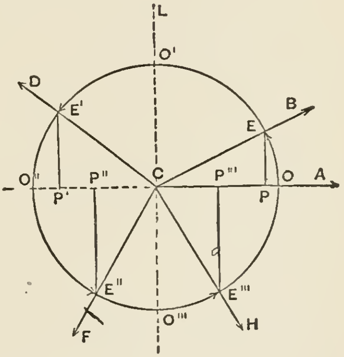
Thus, if $CO = R$,

$$\text{vers } ACB = \frac{PO}{R},$$

$$\text{vers } ACD = \frac{P'O}{R},$$

$$\text{vers } ACF = \frac{P''O}{R},$$

$$\text{vers } ACH = \frac{P'''O}{R}$$



Distances estimated along the initial side in the direction CO are positive. The versed-sine of any positive angle is therefore positive.

41. The versed-sine of any given angle may be determined by making a construction similar to that indicated for the sine. *The quotient of the distance from the foot of the corresponding perpendicular to the origin of the arc, by the length of the radius, will be the required versed-sine.*

The versed-sine of an angle being given, to construct a corresponding angle.

Assume the vertex and initial side, and describe a convenient circle. From the origin lay off upon the initial side, in a negative direction, *a distance equal to the product of the radius by the given versine. Through the point thus determined draw a perpendicular to the initial side. Through the vertex and each of the two points in which this perpendicular intersects the circumference draw a right line; either will be the final side of an angle having the given versine.*

Between 0° and 360° there are always two positive angles which correspond to each possible versine. Since the ends of the subtending arcs of the two angles are always in the same perpendicular to the initial side, it follows that the sum of the angles is always 360°. Hence, denoting any angle by ϕ, we have

$$\text{vers } \phi = \text{vers } (360^\circ - \phi)$$

42. Variation of the Versine as the Angle Changes. The versine varies directly with the distance from the foot of the perpendicular above described to the origin of the subtending arc.

As the angle increases from 0° to 180°, this distance increases from 0 to $2R$; hence the versine increases from 0 to 2

From 180° to 360°, the distance decreases from $2R$ towards 0; hence the versine decreases from 2 towards 0.

If the angle is greater than 360°, we have

$$\text{vers } (n360^\circ + a) = \text{vers } a.$$

2 is the greatest possible value for a versine.

The versine changes most rapidly when the angle is passing through 90° and 270°, and least rapidly when passing through 0° and 180°.

43. *Versines of Negative Angles.* From the definition and figure it is evident that the versine of a negative angle is the same as the versine of an equal positive angle: that is.

$$\text{vers } (-\phi) = \text{vers } \phi.$$

Exercises.

By construction and measurement find the versine of each of the following angles:

$30^\circ, \quad -\frac{\pi}{5}, \quad 60^\circ, \quad -120^\circ, \quad \frac{3\pi}{4}, \quad 240^\circ, \quad 330^\circ, \quad 495^\circ, \quad -930^\circ.$

Construct both positive angles which correspond to each of the following versines :

$\frac{1}{2}$, $\frac{2}{3}$, .4, 1.2, $\frac{5}{4}$, 1.5, $\frac{8}{5}$, 1.9.

44. *The* **coversed-sine** *of an angle is the ratio to the radius, of the distance of the* 90° *point from a right line through the end of the corresponding subtending arc, parallel to the initial side.*

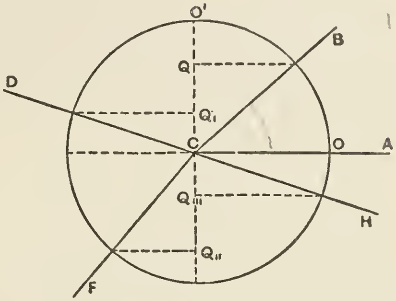
Thus, with radius $CO = R$,

$$\text{covers } ACB = \frac{QO'}{R},$$

$$\text{covers } ACD = \frac{Q_{,}O'}{R},$$

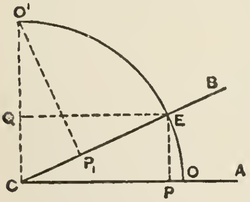
$$\text{covers } ACF = \frac{Q_{,,}O'}{R},$$

$$\text{covers } ACH = \frac{Q_{,,,}O'}{R}.$$



Since the distances QO', $Q_{,}O'$, $Q_{,,}O'$, etc., are always measured towards the 90° point, they are positive, and the coversed-sine of any positive angle is therefore positive.

45. Let ACB be a given angle. Draw $O'P_{,}$ perpendicular to CB. The triangles $CP_{,}O'$ and CQE are equal, and $CQ = CP_{,}$; hence $P_{,}E = QO'$ and versed-sine $BCO' =$ coversed-sine ACB. BCO' is the complement of ACB; and since a similar construction with any given angle, as ϕ, will give a like result, we have covers ϕ = vers $(90° - \phi)$. That is—



The coversed-sine of any angle is equal to the versed-sine of its complement.

The versed-sine and coversed-sine are complementary functions.

46. The coversed-sine of any given angle may be determined by making a construction similar to that indicated for finding the versed-sine, except that in place of a perpendicular to the initial side, a right line parallel to it is drawn through the end of the subtending arc. *The quotient of the distance of the* 90° *point from the parallel line by the length of the radius will be the required coversed-sine.*

An angle corresponding to a given coversed-sine may be constructed in a manner similar to that described for a given versed-sine, except that the 90° point must be used instead of the origin, the right line through the assumed vertex and 90° point, in place of the initial side, and the coversed-sine instead of the versed-sine.

Between 0° and 360° there are always two positive angles which correspond to each possible coversed-sine.

47. Variation of the Coversed-sine as the Angle Changes. The coversine varies directly with the distance of the 90° point from the right line through end of arc, parallel to initial side.

As the angle increases from 0° to 90°, this distance decreases from R to 0; hence the coversine decreases from 1 towards 0.

From 90° to 270°, the distance increases from 0 to $2R$; hence the coversine increases from 0 to 2.

From 270° to 360°, the distance decreases from $2R$ to R; hence the coversine decreases from 2 to 1.

2 is the greatest possible value for a coversine.

If the angle is greater than 360°, we have

$$\text{covers}\,(n360^\circ + a) = \text{covers}\,a.$$

The coversine changes most rapidly when the angle is passing through 0° and 180°, and least rapidly when passing through 90° and 270°.

48. *Coversines of Negative Angles.* From the definition and figure it may be shown that the coversine of a negative angle is equal to 2 minus the coversine of an equal positive angle; that is—

$$\text{covers}\,(-\phi) = 2 - \text{covers}\,\phi = 1 + \sin\phi.$$

Exercises.

By construction and measurement find the coversine of each of the following angles:

30°, -45°, -60°, $\frac{3}{4}\pi$, $-\frac{7}{12}\pi$, 135°, -240°, 300°. 315°.

Construct both positive angles less than 360°, which correspond to each of the following coversines:

$\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, 1.2, .6, 1.5, $\frac{9}{7}$, $\sqrt{2}$, 1.8, $\sqrt{3}$.

49. The following table contains the values of the trigonometric functions of the angles 0°, 90°, 180°, 270°, 360°, and indicates the manner of the variation of each function as the angle increases.

The double sign indicates a change from the upper to the lower as the angle increases. $<$ denotes increasing, and $>$ decreasing algebraically.

TABLE A.

Angles ☞ *Functions*	0°	*to*	90°	*to*	180°	*to*	270°	*to*	360°
Sine...........	∓ 0	$<$	$+1$	$>$	± 0	$>$	-1	$<$	∓ 0
Cosine........	$+1$	$>$	± 0	$>$	-1	$<$	∓ 0	$<$	$+1$
Tangent........	∓ 0	$<$	$\pm \infty$	$<$	∓ 0	$<$	$\pm \infty$	$<$	∓ 0
Cotangent.....	$\mp \infty$	$>$	± 0	$>$	$\mp \infty$	$>$	± 0	$>$	$\mp \infty$
Secant..........	$+1$	$<$	$\pm \infty$	$<$	-1	$>$	$\mp \infty$	$>$	$+1$
Cosecant.......	$\mp \infty$	$>$	$+1$	$<$	$\pm \infty$	$<$	-1	$>$	$\mp \infty$
Versed-sine....	$+0$	$<$	$+1$	$<$	$+2$	$>$	$+1$	$>$	$+0$
Coversed-sine..	$+1$	$>$	$+0$	$<$	$+1$	$<$	$+2$	$>$	$+1$

It should be observed that the tangent always increases, and the cotangent always decreases as the angle increases; also that each function, except versine and coversine, passes through all its possible numerical values in each quadrant separately.

50. Inverse Functions. Angles and their trigonometric functions are mutually dependent. Each function varies with the angle, and an angle varies with each function.

Angles are expressed in terms of their functions; thus an angle whose sine is y is written $\sin^{-1}y$. $\sin\phi = y$ and $\phi = \sin^{-1}y$ are equivalent expressions. Similarly, we have $\cos^{-1}y$, $\tan^{-1}y$, $\text{cotan}^{-1}y$, $\sec^{-1}y$, $\text{cosec}^{-1}y$, $\text{vers}^{-1}y$ and $\text{covers}^{-1}y$. Angles thus expressed are called *inverse trigonometric functions.*

The methods for constructing inverse functions (angles) from given functions have been explained. In case an angle is given in terms of a function of a function, as $\sin^{-1}(\cos 30^\circ)$; determine the $\cos 30^\circ$, and then construct the angle whose sine is equal to the resulting number.

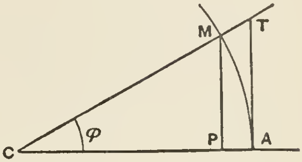
Exercises.

Construct both positive angles less than 360°, in each case given as follows:

$\sin^{-1}\frac{3}{4}$, $\cos^{-1} 7$, $\sin^{-1}(\cos 30^\circ)$, $\cos^{-1}(\sin 30^\circ)$, $\sin^{-1}(\sin 30^\circ)$, $\tan^{-1}(\sin 45^\circ)$, $\cot^{-1}(\cos 60^\circ)$, $\tan^{-1}(\tan -15^\circ)$, $\cot^{-1}(\tan -405^\circ)$, $\sec^{-1}(\cos 60^\circ)$, $\text{cosec}^{-1}\left(\tan\frac{\pi}{4}\right)$, $\sec^{-1}\left(\tan\frac{\pi}{3}\right)$, $\text{cosec}^{-1}(\cot 45^\circ)$, $\text{vers}^{-1}\left(\cos\frac{\pi}{4}\right)$, $\text{covers}^{-1}(\tan 60^\circ)$, $\text{vers}^{-1}(\text{covers} -60^\circ)$, $\text{covers}^{-1}(\sec 45^\circ)$.

CHAPTER III.

ELEMENTARY RELATIONS OF TRIGONOMETRIC FUNCTIONS.



51. Denote any angle, as ACM, by ϕ, and with any radius, as $CA = R$, describe a subtending arc, as AM. At A draw the tangent AT; and from M let fall MP perpendicular to CA.

The right triangle CPM gives

$$\frac{\overline{PM}^2}{R^2} + \frac{\overline{CP}^2}{R^2} = \frac{\overline{CM}^2}{R^2}, \quad \text{or} \quad \mathbf{\sin^2 \phi + \cos^2 \phi = 1}. \quad . \quad . \quad (1)^*$$

Also,

$$\frac{PA}{R} = \frac{CA}{R} - \frac{CP}{R}, \qquad \text{or} \quad \mathbf{vers\ \phi = 1 - \cos \phi}. \quad . \quad . \quad (2)$$

Substituting $\left(\frac{\pi}{2} - \phi\right)$ for ϕ in (2), we have

$$\text{vers}\left(\frac{\pi}{2} - \phi\right) = 1 - \cos\left(\frac{\pi}{2} - \phi\right), \text{ or } \mathbf{covers\ \phi = 1 - \sin \phi}. \quad (3)$$

The similar triangles CPM and CAT give

$$\frac{CP}{R} : \frac{PM}{R} :: \frac{CA}{R} : \frac{AT}{R}, \quad \text{or} \quad \cos \phi : \sin \phi :: 1 : \tan \phi.$$

Hence,

$$\mathbf{\tan \phi = \frac{\sin \phi}{\cos \phi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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* $\sin^2 \phi$, $\cos^2 \phi$, etc., indicate the *square* of the sine, cosine, etc., of ϕ; and are read "sine squared ϕ," etc.

Equations printed in this type should be remembered.

Substituting $\left(\frac{\pi}{2} - \phi\right)$ for ϕ in (4), we have

$$\cot\phi = \frac{\cos\phi}{\sin\phi}. \tag{5}$$

Multiplying (4) and (5), member by member, we have

$$\tan\phi \cot\phi = 1. \tag{6}$$

The triangles *CPM* and *CAT* also give

$$\frac{CP}{R} : \frac{CM}{R} :: \frac{CA}{R} : \frac{CT}{R}, \quad \text{or} \quad \cos\phi : 1 :: 1 : \sec\phi.$$

Hence
$$\sec\phi = \frac{1}{\cos\phi}. \tag{7}$$

Substituting $\left(\frac{\pi}{2} - \phi\right)$ for ϕ in (7), we have

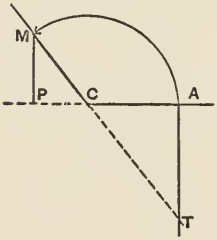
$$\operatorname{cosec}\phi = \frac{1}{\sin\phi}. \tag{8}$$

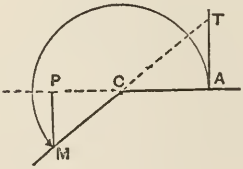
From the right triangle *CAT* we have

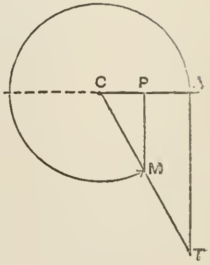
$$\frac{\overline{CT}^2}{R^2} = \frac{\overline{CA}^2}{R^2} + \frac{\overline{AT}^2}{R^2}, \quad \text{or} \quad \sec^2\phi = 1 + \tan^2\phi; \tag{9}$$

whence we obtain
$$\operatorname{cosec}^2\phi = 1 + \cot^2\phi. \tag{10}$$

By applying the foregoing deductions to the following figures the generality of the formulas will become apparent.







Formulas (3), (5), (8) and (10) may be deduced directly from a figure by drawing lines which correspond to the functions which enter them. Let the student do so.

It should be observed (6) that the *tangent* and *cotangent* of any angle are reciprocals of each other; also the *secant* and *cosine* (7); and the *cosecant* and *sine* (8).

The foregoing formulas are of frequent use, and are arranged for convenient reference in

TABLE B.

$\sin^2\phi + \cos^2\phi = 1.$	(1)	$\tan\phi \cot\phi = 1.$	(6)
$\text{vers}\,\phi = 1 - \cos\phi.$	(2)	$\sec\phi = \dfrac{1}{\cos\phi}.$	(7)
$\text{covers}\,\phi = 1 - \sin\phi.$	(3)	$\text{cosec}\,\phi = \dfrac{1}{\sin\phi}.$	(8)
$\tan\phi = \dfrac{\sin\phi}{\cos\phi}.$	(4)	$\sec^2\phi = 1 + \tan^2\phi$	(9)
$\cot\phi = \dfrac{\cos\phi}{\sin\phi}.$	(5)	$\text{cosec}^2\phi = 1 + \cot^2\phi.$	(10)

52. *To express any trigonometric function in terms of any other of the same angle.*

If one of the above formulas contains the required and given functions only, solve with respect to the required function

Thus, to express $\sin\phi$ in terms of $\cos\phi$, from (1),

$$\sin\phi = \pm\sqrt{1 - \cos^2\phi}.$$

If none contain these two functions only, select a group of independent formulas containing beside these two functions, one or more auxiliary ones. There should be one more equation than there are auxiliary functions. Combine and eliminate all the auxiliary functions, and solve the resulting equation with respect to the required function. Thus, to express vers ϕ in terms of covers ϕ, take (1), (2) and (3) with the two auxiliary functions sin ϕ and cos ϕ. Eliminating these by substitution, we have

$$(1 - \text{vers}\ \phi)^2 + (1 - \text{covers}\ \phi)^2 = 1;$$

$$\therefore\ \text{vers}\ \phi = 1 \pm \sqrt{\text{covers}\ \phi\ (2 - \text{covers}\ \phi)}.$$

Exercises.

1. Given $\sin \phi = \frac{1}{2}$, find cos ϕ and covers ϕ.

Ans. $\cos \phi = \pm \sqrt{\frac{3}{4}} = \pm \frac{1}{2}\sqrt{3}$, $\text{covers}\ \phi = \frac{1}{2}$.

2. Given $\text{vers}\ \phi = 1.5$, find cos ϕ and sec ϕ.

Ans. $\cos \phi = -\frac{1}{2}$, $\sec \phi = -2$.

3. Given $\tan \vartheta = 1$, find sin ϑ and cos ϑ.

Ans. $\sin \vartheta = \pm \sqrt{\frac{1}{2}}$, $\cos \vartheta = \pm \sqrt{\frac{1}{2}}$.

4. Given $\cot A = \frac{1}{3}$, show that

$\tan A = 3$, $\sec A = \pm \sqrt{10} = 3.162 -$,

$\text{cosec}\ A = \pm \frac{1}{3}\sqrt{10}$, $\cos A = \pm \sqrt{\frac{1}{10}} = 0.3162 +$,

$\sin A = \pm 3\sqrt{\frac{1}{10}}$, $\text{vers}\ A = 1 \mp \sqrt{\frac{1}{10}} = 0.6938$ or 1.3162,

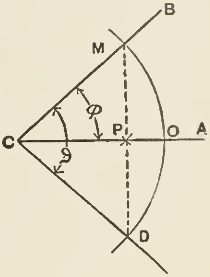
$\text{covers}\ A = 1 \mp 3\sqrt{\frac{1}{10}} = 0.0514$ or 1.9486.

5. Express $\sin\phi$ separately in terms of each of the other trigonometric functions.

6. Same for $\cos\phi$.

7. Same for $\tan\phi$.

53. *Chord and Height of Arc.* Let $\phi = ACB$ be any angle less than 180°. Produce the arc at O, making DO equal to the intercepted arc OM. Draw CD and DM. The radius CO bisects the arc DOM and is perpendicular to the chord DM.



$$DCM = 2\phi \qquad \text{and} \qquad DM = 2PM.$$

$$\sin\phi = \frac{PM}{R}.$$

Hence

$$\sin\phi = \frac{\frac{1}{2}DM}{R} = \frac{\frac{1}{2}\text{ chord of }2\phi R}{R}. \quad . \quad . \quad . \quad . \quad (11)$$

That is—

The sine of any angle less than 180° *is equal to the ratio of half the chord of twice the subtending arc to the corresponding radius.*

Thus chord for 60° $= R \quad \therefore \sin 30° = \frac{\frac{1}{2}R}{R} = \frac{1}{2}$.

Examples.

1. Given chord for $90° = R\sqrt{2}$, find $\sin 45°$.

2. Given chord for $36° = \frac{-R + R\sqrt{5}}{2}$, find $\sin 18°$.

To find the chord of any arc, put ϑ for 2ϕ in (11) and reduce, giving

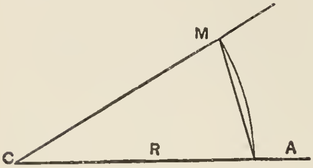
$$\text{chord for } \vartheta = 2R \sin \tfrac{1}{2}\vartheta. \qquad (12)$$

But arc $DOM = R\vartheta$; hence

$$\text{chord of arc } R\vartheta = 2R \sin \tfrac{1}{2}\vartheta.$$

That is—

The chord of any arc is equal to twice the product of the radius by the sine of half the angle at the centre.



To construct an angle knowing the sine of half the angle. Assume the initial side, and with a radius $CA = R = 5$ units of scale, describe an arc, as AM. With a radius equal to the product of the given sine by 10, and with A as a centre, describe an arc cutting the first arc at some point M. ACM will be the required angle.

Examples.

1. Given $\sin 11^\circ = 0.19$, construct 22°.

2. Given $\sin 24^\circ 50' = 0.42$, construct $49^\circ 40'$.

To find the height of any arc above its chord (see figure), we have

$$PO = \text{height of arc } R\vartheta \text{ from chord } DM.$$

But $\qquad PO = R \text{ vers } \phi, \qquad \text{and} \qquad \phi = \tfrac{1}{2}\vartheta.$

$$\therefore \quad PO = R \text{ vers } \tfrac{1}{2}\vartheta. \qquad (13)$$

That is—

The height of any arc is equal to the product of the radius by the versed-sine of half the angle at the centre.

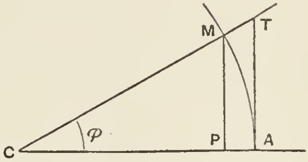
54 *Values of the functions of* 30°, 45°, 60° *and* 18°.

Since chord of 60° = R, (11) gives

$$\sin 30° = \tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Hence

$$\cos 60° = \tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$



In the right triangle ACT, if angle $ACT = 45°$

$AT = CA = R$ and $\tan 45° = 1$. . (16)

From (14), (15) and (16), the values of all the trigonometric functions of 30°, 45°, 60° are computed and arranged for reference in

TABLE C.

☟ *Angles* *Functions* ☞	sin	cos	vers	covers	tan	cot	sec	cosec
30°	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$1 - \frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$1 - \frac{1}{2}\sqrt{2}$	$1 - \frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$1 - \frac{1}{2}\sqrt{3}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$

Let the student verify the values in Table C, also the following:

$$\sin 18° = \frac{-1 + \sqrt{5}}{4}, \qquad \cos 18° = \sqrt{\frac{5 + \sqrt{5}}{8}},$$

$$\tan 18^\circ = \sqrt{1 - \tfrac{2}{5}\sqrt{5}}, \qquad \cot 18^\circ = \sqrt{5 + 2\sqrt{5}}.$$

$$\sec 18^\circ = \sqrt{2 - \tfrac{2}{5}\sqrt{5}}, \qquad \operatorname{cosec} 18^\circ = 1 + \sqrt{5}.$$

55. Reduction to First Revolution. Negative angles, and positive angles of more than one revolution, may be reduced to the first revolution by the arbitrary addition or subtraction of a suitable number of complete revolutions. This will not alter the trigonometric functions, since the position of each side of the angle is the same after as before the change. Thus,

$$\sin(-500^\circ) = \sin[-500^\circ + 720^\circ] = \sin 220^\circ,$$

$$\cos(400^\circ) = \cos[400^\circ - 360^\circ] = \cos 40^\circ,$$

$$\tan(-\pi) = \tan[-\pi + 2\pi] = \tan \pi,$$

$$\operatorname{vers}\left(\frac{9\pi}{2}\right) = \operatorname{vers}\left[\frac{9\pi}{2} - 4\pi\right] = \operatorname{vers}\frac{\pi}{2},$$

$$\operatorname{cosec}(-1) = \operatorname{cosec}[-1 + 6.2832] = \operatorname{cosec} 5.2832.$$

Examples.

Reduce to the first revolution the following trigonometric functions:

$\sin 1091^\circ, \quad -30^\circ, \quad -\frac{3\pi}{2};$

$\cos 500^\circ, \quad -250^\circ, \quad \frac{7\pi}{4};$

$\tan 1871^\circ\, 15'\, 25'', \quad 71\pi;$

$\cot -1590^\circ\, 21'\, 37'', \quad -\frac{91\pi}{3};$

$\sec -9.2832, \quad 7.2832;$

$\operatorname{covers} -3; \quad 14.5664.$

56. Reduction to First Quadrant. If ϕ denotes any positive angle less than 90°, we may express angles

$$\text{from } 90^\circ \text{ to } 180^\circ \quad \text{by} \quad 180^\circ - \phi,$$

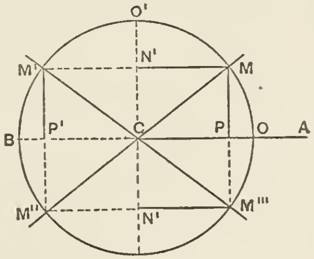
$$\text{from } 180^\circ \text{ to } 270^\circ \quad \text{by} \quad 180^\circ + \phi,$$

$$\text{from } 270^\circ \text{ to } 360^\circ \quad \text{by} \quad 360^\circ - \phi.$$

Let $\phi = ACM$, $180^\circ - \phi = ACM'$,

$$180^\circ + \phi = ACM'' \quad \text{and} \quad 360^\circ - \phi = ACM'''.$$

With any radius R draw the circumference $OMM'M''M'''$. Draw MP, $M'P'$, $M''P'$ and $M'''P$ each perpendicular to CA.



Since $ACM' = 180^\circ - \phi$,

$$M'CB = \phi = ACM.$$

Hence, the right triangles CPM and $CP'M'$ are equal in all their parts, and regarding the signs of the lines, we have

$$\frac{P'M'}{R} = \frac{PM}{R}, \quad \text{and} \quad \frac{CP'}{R} = \frac{-CP}{R};$$

giving $\sin(180^\circ - \phi) = \sin\phi$, $\cos(180^\circ - \phi) = -\cos\phi$.

In like manner, we have

$$\frac{P'M''}{R} = \frac{-PM}{R}, \quad \text{and} \quad \frac{CP'}{R} = \frac{-CP}{R};$$

giving $\sin(180^\circ + \phi) = -\sin\phi$, $\cos(180^\circ + \phi) = -\cos\phi$.

Also, $\dfrac{PM'''}{R} = \dfrac{-PM}{R}$, and $\dfrac{CP}{R} = \dfrac{CP}{R}$;

giving $\sin(360° - \phi) = -\sin\phi$, $\cos(360° - \phi) = \cos\phi$.

Substituting in above $90° - \phi$, which is positive and less than $90°$, for ϕ, we have

$$\sin(90° + \phi) = \cos\phi, \qquad \cos(90° + \phi) = -\sin\phi;$$

$$\sin(270° - \phi) = -\cos\phi, \qquad \cos(270° - \phi) = -\sin\phi;$$

$$\sin(270° + \phi) = -\cos\phi, \qquad \cos(270° + \phi) = \sin\phi.$$

The expressions for $\sin(180° - \phi)$ and $\cos(180° - \phi)$ substituted in the formulas for tangent, cotangent, secant, cosecant, versed-sine and coversed-sine, Table B, give

$$\tan(180° - \phi) = \frac{\sin\phi}{-\cos\phi} = -\tan\phi,$$

$$\cot(180° - \phi) = \frac{-\cos\phi}{\sin\phi} = -\cot\phi,$$

$$\sec(180° - \phi) = \frac{1}{-\cos\phi} = -\sec\phi,$$

$$\operatorname{cosec}(180° - \phi) = \frac{1}{\sin\phi} = \operatorname{cosec}\phi,$$

$$\operatorname{vers}(180° - \phi) = 1 - (-\cos\phi) = 2 - [1 - \cos\phi] = 2 - \operatorname{vers}\phi,$$

$$\operatorname{covers}(180° - \phi) = 1 - \sin\phi = \operatorname{covers}\phi.$$

In like manner, we may find expressions for the functions of $(180° + \phi)$, $(360° - \phi)$, $(90° + \phi)$, $(270° - \phi)$ and $(270° + \phi)$ in terms of those of ϕ.

The results are arranged in

TABLE D.

QUADRANTS.	Second	Third	Fourth	Second	Third	Fourth
Funct. *Angles*	$180° - \phi$	$180° + \phi$	$360° - \phi$	$90° + \phi$	$270° - \phi$	$270° + \phi$
sin	$\sin\phi$	$-\sin\phi$	$-\sin\phi$	$\cos\phi$	$-\cos\phi$	$-\cos\phi$
cos	$-\cos\phi$	$-\cos\phi$	$\cos\phi$	$-\sin\phi$	$-\sin\phi$	$\sin\phi$
tan	$-\tan\phi$	$\tan\phi$	$-\tan\phi$	$-\cot\phi$	$\cot\phi$	$-\cot\phi$
cot	$-\cot\phi$	$\cot\phi$	$-\cot\phi$	$-\tan\phi$	$\tan\phi$	$-\tan\phi$
sec	$-\sec\phi$	$-\sec\phi$	$\sec\phi$	$-\text{cosec}\,\phi$	$-\text{cosec}\,\phi$	$\text{cosec}\,\phi$
cosec	$\text{cosec}\,\phi$	$-\text{cosec}\,\phi$	$-\text{cosec}\,\phi$	$\sec\phi$	$-\sec\phi$	$-\sec\phi$
vers	$2 - \text{vers}\,\phi$	$2 - \text{vers}\,\phi$	$\text{vers}\,\phi$	$2 - \text{covers}\,\phi$	$2 - \text{covers}\,\phi$	$\text{covers}\,\phi$
covers	$\text{covers}\,\phi$	$2 - \text{covers}\,\phi$	$2 - \text{covers}\,\phi$	$\text{vers}\,\phi$	$2 - \text{vers}\,\phi$	$2 - \text{vers}\,\phi$

This table contains expressions by which the functions of all angles from 90° to 360° may be reduced to the first quadrant, i.e. expressed in terms of those of a positive angle less than 90°.

The first column contains the functions to be reduced, and the expression for each function is opposite to it in the column of the given angle.

This table, or any part of it, may be easily rewritten by observing that the functions in the columns which relate to 180° or 360° have the same names as those in the first column; and that the functions in the columns of angles which relate to 90° or 270° have complementary names. Thus—

$$\tan(180° + \phi) = \tan\phi, \qquad \text{covers}\,(360° - \phi) = 2 - \text{covers}\,\phi,$$

$$\sin(90° + \phi) = \cos\phi, \qquad \text{cosec}\,(270° + \phi) = -\sec\phi.$$

The student will readily give the proper algebraic sign to any function by noting the quadrant of the angle.

Thus, $\text{vers}\,(180° - \phi) = 2 - \text{vers}\,\phi$

$$\sec(180° - \phi) = -\sec\phi$$

The angle $180° - \phi$ being in the second quadrant, its versed-sine is greater than unity, and its secant negative.

Let it be required to reduce to the first quadrant sin 115° and tan 195°.

$$\sin 115° = \sin (90° + 25°) = \cos 25°;$$

otherwise, $$\sin 115° = \sin (180° - 65°) = \sin 65°.$$

$$\tan 195° = \tan (180° + 15°) = \tan 15°;$$

or, $$\tan 195° = \tan (270° - 75°) = \cot 75° = \tan 15°.$$

It may be necessary to reduce to the first revolution, and then to the first quadrant. Thus—

$$\cos 410° = \cos (410° - 360°) = \cos 50° = \sin 40°;$$
$$\operatorname{vers} 520° = \operatorname{vers} 160° = \operatorname{vers} (180° - 20°) = 2 - \operatorname{vers} 20° = 2 - \operatorname{covers} 70°;$$
$$\cot (-10°) = \cot (-10° + 360°) = \cot (360° - 10°) = -\cot 10° = -\tan 80°;$$
$$\sin (-135°) = \sin 225° = \sin (180° + 45°) = -\sin 45° = -\cos 45°.$$

Exercises.

Reduce to the first quadrant in two ways each of the following trigonometric functions:

sin 110°,	240°,	290°,	− 75°;
cos 115°,	235°,	295°,	391°;
tan 125°,	245°,	300°,	750°;
cot 120°,	220°,	320°,	− 750°;
sec 135°,	210°,	330°,	1834°;
cosec 130°,	205°,	310°,	− 962°;
vers 140°,	215°.	305°,	9320°;
covers 150°	225°,	315°,	694°;
sin 359° 17′ 21″;			
cot − 420° 45′ 32″;			
sec 550° 53′ 08″.			

Computers usually prefer the method which gives the fractional part of a degree the same after as before reduction.

57. *Functions of Negative Angles in Terms of those of Equal Positive Angles.* Let ϕ be any positive angle. It has been shown that

$$\sin (-\phi) = -\sin \phi, \quad \text{and} \quad \cos (-\phi) = \cos \phi,$$

which expressions are identical with those already deduced for $\sin(360^\circ - \phi)$ and $\cos(360^\circ - \phi)$, and if substituted in (4), (5), (7), (8), (2) and (3), will give the expressions arranged under $(360^\circ - \phi)$ in Table D.* This agreement should exist since $-\phi$ when reduced to the first revolution is $(n360^\circ - \phi)$, n being a suitable positive integer.

Functions of negative angles are frequently reduced to the first quadrant by passing to those of their numerically equal positive angles, and then reducing. Thus—

$$\tan(-163^\circ) = -\tan 163^\circ = -\tan(90^\circ + 73^\circ) = \cot 73^\circ = \tan 17^\circ;$$
$$\cos(-275^\circ) = \cos 275^\circ = \cos(270^\circ + 5^\circ) = \sin 5^\circ = \cos 85^\circ;$$
$$\sec(-190^\circ) = \sec 190^\circ = \sec(180^\circ + 10^\circ) = -\sec 10^\circ = -\operatorname{cosec} 80^\circ.$$

Examples.

Express the following functions of negative angles in terms of those of numerically equal-positive angles, and then reduce to the first quadrant:

1. $\sin(-240^\circ)$. *Ans.* $-\sin 240^\circ = \sin 60^\circ = \cos 30^\circ$.
2. $\operatorname{covers}(-280^\circ)$. *Ans.* $2 - \operatorname{covers} 280^\circ = \operatorname{vers} 10^\circ = \operatorname{covers} 80^\circ$.
3. $\tan(-550^\circ)$. *Ans.* $-\tan 550^\circ = -\tan 10^\circ = -\cot 80^\circ$.
4. $\cos\left(-\frac{3\pi}{2}\right)$. *Ans.* $\cos\frac{3\pi}{2} = -\cos\frac{\pi}{2}$.
5. $\operatorname{cosec}(-2)$. *Ans.* $-\operatorname{cosec} 2 = -\sec(0.4292)$
$= -\operatorname{cosec}(1.1416)$.

58. The methods of reduction to the first quadrant may be reversed to find such angles as have a specified function given in terms of the same or the complementary function of an angle in the first quadrant.

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*Since ϕ is of unlimited value in this discussion, the results agreeing with those in one column of Table D establish the general truth of that column. The general truth of the other columns may also be proved.

Thus, if $\sin \phi = - \cos 20^\circ$, ϕ must be in the third or fourth quadrant since its sine is negative; and since the functions are complementary the question must relate to 90° or 270°.

Hence, $\phi = 270^\circ \mp 20^\circ = 250^\circ$ or 290°.

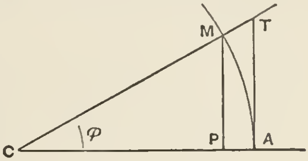
Other values may be found by the arbitrary addition or subtraction of integral multiples of 360°.

Examples.

Find all positive values of $\phi < 360^\circ$:

1. When $\sin \phi = \sin 10^\circ$. *Ans.* 10° and 170°.
2. When $\cos \phi = - \sin 20^\circ$. *Ans.* 110° and 250°.
3. When $\cot \phi = - \cot 40^\circ$. *Ans.* 140° and 320°.
4. When $\sec \phi = \sec 75^\circ$. *Ans.* 75° and 285°.
5. When $\sin \phi = - \frac{1}{2}$. *Ans.* 210° and 330°.
6. When $\cos \phi = \frac{1}{2}$. *Ans.* 60° and 300°.
7. When $\tan \phi = - 1$. *Ans.* 135° and 315°.
8. $\text{vers}\, \phi = \text{vers}\, 34^\circ$. *Ans.* 34° and 326°.
9. $\text{vers}\, \phi = 2 - \text{covers}\, 40^\circ$. *Ans.* 130° and 230°.
10. $\tan 3\phi = \tan 45^\circ$. In this example we have for 3ϕ the positive values 45°, 225° [45° + 360°], [225° + 360°], etc. In order to obtain all positive values of ϕ less than 360°, all positive values of 3ϕ less than $3 \times 360^\circ$ must be separately divided by 3, giving 15°, 75°, 135°, 195°, 255°, 315°.
11. $\sin 2\phi = \sin 15^\circ$. *Ans.* $7\frac{1}{2}^\circ$, $82\frac{1}{2}^\circ$, $187\frac{1}{2}^\circ$, $262\frac{1}{2}^\circ$.
12. $\cot 3\phi = - \tan 75^\circ$. *Ans.* 55°, 115°, 175°, 235°, 295°, 355°.
13. $\text{covers}\, 2\phi = 2 - \text{covers}\, 10^\circ$. *Ans.* 95°, 175°, 275°, 355°.
14. $\cos 2\phi = \frac{1}{2}$.
15. $\tan 3\phi = - 1$.
16. Given $\sin 30^\circ = \frac{1}{2}$, find the sine and cosine of each angle less than 360°, which is an integral multiple of 30°.
17. If the sine and cosine of 0°, 10°, 20°, 30° and 40° are known, show how to find the sine and cosine of every angle less than 360°, which is an integral multiple of 10°.

59. *The radian measure of any acute angle is greater than its sine, and less than its tangent.* Let ϕ be any acute angle, as ACT. From C as a centre, with $CA = R$, draw the intercepted arc AM. Draw AT tangent to it at A, and meeting CM at T. From M let fall the perpendicular MP on CA.



Then $\quad$ sector $CAM <$ triangle CAT;

or, $\quad \frac{1}{2}R \times \text{arc } AM < \frac{1}{2}R \times AT$.

Hence, $\quad$ arc $AM < AT$.

But MP is the shortest distance from CA to M. Hence—

$$\frac{PM}{R} < \frac{\text{arc } AM}{R} < \frac{AT}{R};$$

or,
$$\sin\phi < \phi < \tan\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

60. *Unity is the limit of the ratio of an angle to its sine, of an angle to its tangent, and of the tangent to the sine, as the angle approaches zero.*

From (17) we have $\frac{\tan\phi}{\sin\phi} > \frac{\phi}{\sin\phi} > 1$; but $\frac{\tan\phi}{\sin\phi} = \frac{1}{\cos\phi}$.

$$\underset{\phi \to 0}{\text{Limit}}\left[\frac{\tan\phi}{\sin\phi}\right] = \underset{\phi\to 0}{\text{limit}}\left[\frac{1}{\cos\phi}\right] = 1, \quad \therefore \quad \underset{\phi\to 0}{\text{limit}}\left[\frac{\phi}{\sin\phi}\right] = 1.$$

Since, $\frac{\phi}{\sin\phi}$ is always between unity and $\frac{\tan\phi}{\sin\phi}$.

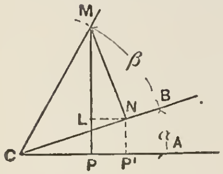
Also, $\quad 1 > \frac{\phi}{\tan\phi} > \frac{\sin\phi}{\tan\phi}$, and $\underset{\phi\to 0}{\text{limit}}\left[\frac{\sin\phi}{\tan\phi}\right] = 1.$

Hence,
$$\underset{\phi\to 0}{\text{limit}}\left[\frac{\phi}{\tan\phi}\right] = 1.$$

CHAPTER IV.

FUNCTIONS OF TWO OR MORE ANGLES.

61. *Sine of the Sum of Two Angles.* Let $\alpha = ACB$, and $\beta = BCM$. Then $\alpha + \beta = ACM$. From C as a centre, with $CA = R$, draw the arc ABM. Draw PM and NM perpendicular to CA and CB, respectively. Through N draw $P'N$ perpendicular, and LN parallel, to CA.



Taking CN as an auxiliary radius, we have

$$\sin\alpha = \frac{P'N}{CN}, \text{ and } \cos\alpha = \frac{CP'}{CN}.$$

We also have

$$\sin\beta = \frac{NM}{R}, \quad \cos\beta = \frac{CN}{R},$$

and

$$\sin(\alpha+\beta) = \frac{PM}{R} = \frac{PL}{R} + \frac{LM}{R},$$

$$\frac{PL}{R} = \frac{P'N}{R} = \frac{P'N}{CN}\cdot\frac{CN}{R} = \sin\alpha\cos\beta,$$

$$\frac{LM}{R} = \frac{LM}{NM}\cdot\frac{NM}{R}.$$

The triangle MLN has its sides, respectively, perpendicular to those of $CP'N$. Hence they are similar, and $\frac{LM}{NM} = \frac{CP'}{CN} = \cos\alpha$.

Hence,

$$\frac{LM}{R} = \cos\alpha\sin\beta.$$

Substituting for $\frac{PL}{R}$ and $\frac{LM}{R}$ the foregoing expressions, we have

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \quad . \quad . \quad . \quad (28)$$

The generality of this formula is made apparent by applying the foregoing deduction to the following figures:

Fig. 1.

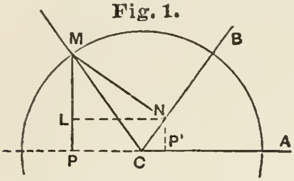


Fig. 2.

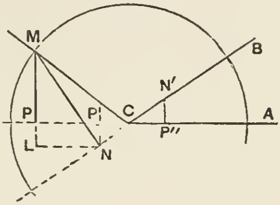


Fig. 3.

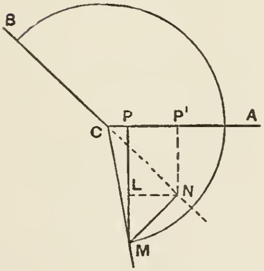


Fig. 4.

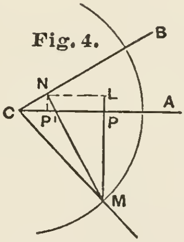
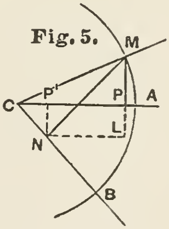


Fig. 5.



NM is positive when M is to the left of CB, as in all these figures except Fig. 4.

To verify the relations,

$$\sin\alpha = \frac{P'N}{CN}, \qquad \cos\alpha = \frac{CP'}{CN},$$

when CN is negative. Take an auxiliary point N', so that $CN' = -CN$ (Fig. 2). Then

$$\sin\alpha = \frac{P''N'}{CN'} = \frac{-P'N}{-CN} = \frac{P'N}{CN},$$

$$\cos\alpha = \frac{CP''}{CN'} = \frac{-CP'}{-CN} = \frac{CP'}{CN}.$$

62. *Functions of* $(\alpha \pm \beta)$. Since (28) is true for all values of α and β, we may replace β by $-\beta$, giving

$$\sin(\alpha - \beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta).$$

But $\sin(-\beta) = -\sin\beta$, and $\cos(-\beta) = \cos\beta$.

Hence,

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta; \quad . \quad . \quad . \quad (29)$$

in which putting $90° - \alpha$ for α, we have

$$\sin(90° - \alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

or

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta; \quad . \quad . \quad . \quad (30)$$

in which putting $-\beta$ for β, gives

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \quad . \quad . \quad (31)$$

From (28) and (29) we have

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta. \quad . \quad . \quad (32)$$

From (30) and (31),

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \quad . \quad . \quad . \quad (33)$$

tan $(\alpha \pm \beta)$, cot $(\alpha \pm \beta)$, sec $(\alpha \pm \beta)$, cosec $(\alpha \pm \beta)$, vers $(\alpha \pm \beta)$ and covers $(\alpha \pm \beta)$ may be expressed in terms of sin $(\alpha \pm \beta)$ and cos $(\alpha \pm \beta)$, and then by (32) and (33) expressed in terms of functions of α and β. Thus—

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}. \quad . \quad (34)$$

$$\cot(\alpha \pm \beta) = \frac{\cos(\alpha \pm \beta)}{\sin(\alpha \pm \beta)} = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}. \quad . \quad (35)$$

Examples.

	Given.	*Required.*
1.	$\sin 30^\circ = 0.50$, $\cos 30^\circ = 0.87$, $\sin 45^\circ = 0.71$, $\cos 45^\circ = 0.71$,	$\sin 15^\circ$, $\sin 75^\circ$.
2.	Same.	$\cos 15^\circ$, $\cos 75^\circ$.
3.	$\tan 30^\circ = 0.58$, $\tan 45^\circ = 1.00$,	$\tan 15^\circ$, $\tan 75^\circ$.
4.	$\cot 30^\circ = 1.73$, $\cot 45^\circ = 1.00$,	$\cot 15^\circ$, $\cot 75^\circ$.
5.	$\tan x = \frac{1}{2}$, $\tan y = \frac{1}{3}$,	$\tan(x+y)$, $\tan(x-y)$.
6.	$\cot x = 3$, $\cot y = 2$,	$\cot(x+y)$, $\cot(x-y)$.

The student should verify the formulas.

7. $\cos(45^\circ \pm \phi) = \sin(45^\circ \mp \phi) = \dfrac{\cos\phi \mp \sin\phi}{\sqrt{2}}$.

8. $\tan(45^\circ \pm x) = \dfrac{1 \pm \tan x}{1 \mp \tan x}$.

63. Functions of the sum or difference of any number of angles may be expressed in terms of the functions of those angles by the successive application of the preceding formulas. Thus—

$$\begin{aligned} \sin(\alpha+\beta+\gamma) &= \sin(\alpha+[\beta+\gamma]) \\ &= \sin\alpha\cos(\beta+\gamma)+\cos\alpha\sin(\beta+\gamma) \\ &= \left.\begin{array}{l}\cos\beta\cos\gamma \\ -\sin\beta\sin\gamma\end{array}\right| \sin\alpha + \left.\begin{array}{r}\sin\beta\cos\gamma \\ +\cos\beta\sin\gamma\end{array}\right| \cos\alpha \end{aligned}$$

or

$$\begin{aligned} \sin(\alpha+\beta+\gamma) &= \cos\alpha\cos\beta\sin\gamma+\cos\alpha\sin\beta\cos\gamma \\ &\quad +\sin\alpha\cos\beta\cos\gamma-\sin\alpha\sin\beta\sin\gamma. \end{aligned}$$

Let the student verify the following formulas:

$$\cos(\alpha+\beta+\gamma) = \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma$$
$$- \sin\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma.$$

$$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\alpha\tan\gamma-\tan\beta\tan\gamma}$$

64. *Functions of* 2α. Making $\beta = \alpha$ in (28) and (30), we have

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \ldots\ldots \quad (36)$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha; \quad \ldots\ldots \quad (37)$$

or by (1),

$$\cos 2\alpha = 1 - 2\sin^2\alpha, \quad \ldots\ldots \quad (38)$$

and

$$\cos 2\alpha = 2\cos^2\alpha - 1. \quad \ldots\ldots \quad (39)$$

Similarly from (34) and (35), using the upper sign,

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}, \quad \ldots\ldots \quad (40)$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha}, \quad \ldots\ldots \quad (41)$$

or

$$\cot 2\alpha = \tfrac{1}{2}\cot\alpha - \tfrac{1}{2}\tan\alpha. \quad \ldots \quad (42)$$

From (2) we have

$$\operatorname{vers} 2\alpha = 1 - \cos 2\alpha = 1 - [1 - 2\sin^2\alpha],$$

or

$$\operatorname{vers} 2\alpha = 2\sin^2\alpha. \quad \ldots\ldots \quad (43)$$

In like manner,

$$\operatorname{covers} 2\alpha = 1 - 2\sin\alpha\cos\alpha = 2\sin^2\tfrac{1}{2}(90^\circ - 2\alpha),$$

$$\sec 2\alpha = \frac{1}{\cos^2\alpha - \sin^2\alpha},$$

$$\operatorname{cosec} 2\alpha = \frac{1}{2\sin\alpha\cos\alpha} = \tfrac{1}{2}\sec\alpha\operatorname{cosec}\alpha.$$

Examples.

	Given.	*Required.*
1.	$\sin 15^\circ = 0.259$, $\cos 15^\circ = 0.966$,	$\sin 30^\circ$, $\cos 30^\circ$.
2.	$\sin 75^\circ = 0.966$, $\cos 75^\circ = 0.259$,	$\sin 150^\circ$, $\cos 150^\circ$.
3.	$\tan 15^\circ = 0.268$,	$\tan 30^\circ$.
4.	$\tan 75^\circ = 3.732$,	$\tan 150^\circ$.
5.	$\cot 15^\circ = 3.732$,	$\cot 30^\circ$.
6.	$\tan x = \frac{1}{3}$,	$\tan 2x$.
7.	$\tan y = \frac{1}{2}$,	$\tan 2y$.

65. *Functions of* $\frac{\vartheta}{2}$. Put $\vartheta = 2\alpha$ in (38), giving, after transposing,

$$2 \sin^2 \frac{\vartheta}{2} = 1 - \cos \vartheta, \tag{44}$$

and

$$\sin \frac{\vartheta}{2} = \pm \sqrt{\frac{1 - \cos \vartheta}{2}}. \tag{45}$$

In like manner, from (39),

$$2 \cos^2 \frac{\vartheta}{2} = 1 + \cos \vartheta, \tag{46}$$

and

$$\cos \frac{\vartheta}{2} = \pm \sqrt{\frac{1 + \cos \vartheta}{2}}. \tag{47}$$

These values in (4) and (5) give

$$\tan \frac{\vartheta}{2} = \pm \sqrt{\frac{1 - \cos \vartheta}{1 + \cos \vartheta}}. \tag{48}$$

$$\cot \frac{\vartheta}{2} = \pm \sqrt{\frac{1 + \cos \vartheta}{1 - \cos \vartheta}}. \tag{49}$$

In (29) put $\alpha = \vartheta$ and $\beta = \frac{\vartheta}{2}$, then $\alpha - \beta = \frac{\vartheta}{2}$, and we have

$$\sin\frac{\vartheta}{2} = \sin\vartheta\cos\frac{\vartheta}{2} - \cos\vartheta\sin\frac{\vartheta}{2}.$$

Divide by $\cos\frac{\vartheta}{2}$ and reduce by (4), giving

$$\tan\frac{\vartheta}{2} = \sin\vartheta - \cos\vartheta\tan\frac{\vartheta}{2};$$

$$\therefore \tan\frac{\vartheta}{2} = \frac{\sin\vartheta}{1+\cos\vartheta}. \quad \ldots\ldots \quad (50)$$

In like manner, from (31) we have

$$\cos\frac{\vartheta}{2} = \cos\vartheta\cos\frac{\vartheta}{2} + \sin\vartheta\sin\frac{\vartheta}{2},$$

$$1 = \cos\vartheta + \sin\vartheta\tan\frac{\vartheta}{2};$$

$$\therefore \tan\frac{\vartheta}{2} = \frac{1-\cos\vartheta}{\sin\vartheta}. \quad \ldots\ldots \quad (51)$$

(48) may also be found from (50) or (51) by reduction. Thus from (50),

$$\tan\frac{\vartheta}{2} = \frac{\pm\sqrt{1-\cos^2\vartheta}}{1+\cos\vartheta} = \pm\sqrt{\frac{1-\cos^2\vartheta}{(1+\cos\vartheta)^2}} = \pm\sqrt{\frac{1-\cos\vartheta}{1+\cos\vartheta}}.$$

Additional expressions for $\cot\frac{\vartheta}{2}$ may be found by taking the reciprocals of (50) and (51).

Expressions for vers $\frac{\vartheta}{2}$, covers $\frac{\vartheta}{2}$, sec $\frac{\vartheta}{2}$ and cosec $\frac{\vartheta}{2}$ are little used. They may be easily found from (45) and (47). Thus,

$$\operatorname{vers}\frac{\vartheta}{2} = 1 - \cos\frac{\vartheta}{2} = 1 \mp \sqrt{\frac{1+\cos\vartheta}{2}},$$

or

$$\operatorname{vers}\frac{\vartheta}{2} = 1 \mp \sqrt{\frac{2-\operatorname{vers}\vartheta}{2}}.$$

In like manner,

$$\operatorname{covers}\frac{\vartheta}{2} = 1 \mp \sqrt{\frac{\operatorname{vers}\vartheta}{2}};$$

$$\sec\frac{\vartheta}{2} = \frac{1}{\pm\sqrt{\frac{1+\cos\vartheta}{2}}} = \pm\sqrt{\frac{2}{1+\cos\vartheta}} = \pm\sqrt{\frac{2\sec\upsilon}{\sec\vartheta+1}},$$

$$\operatorname{cosec}\frac{\vartheta}{2} = \pm\sqrt{\frac{2}{1-\cos\vartheta}} = \pm\sqrt{\frac{2\sec\vartheta}{\sec\vartheta-1}}.$$

Examples.

	Given.	*Required.*
1.	$\sin 45^\circ = \sqrt{\frac{1}{2}} = \cos 45^\circ$,	$\sin 22\frac{1}{2}^\circ$, $\cos 22\frac{1}{2}^\circ$.
2.	$\tan 45^\circ = 1 = \cot 45^\circ$,	$\tan 22\frac{1}{2}^\circ$, $\cot 22\frac{1}{2}^\circ$.
3.	$\sec 45^\circ = \sqrt{2}$,	$\sec 22\frac{1}{2}^\circ$, $\operatorname{cosec} 22\frac{1}{2}^\circ$.
4.	$\sin 30^\circ = \frac{1}{2}$,	functions of 15°.
5.	$\sin 150^\circ = \frac{1}{2}$,	functions of 75°.

Prove the following relations:

6. $\sin\left(45^\circ \pm \frac{\vartheta}{2}\right) = \frac{1}{2}(\sqrt{1+\cos\vartheta} \pm \sqrt{1-\cos\vartheta})$.

7. $\cos\left(45^\circ \pm \frac{\vartheta}{2}\right) = \frac{1}{2}(\sqrt{1+\cos\vartheta} \mp \sqrt{1-\cos\vartheta})$.

8. $\operatorname{vers}\vartheta = \sin\vartheta\tan\frac{\vartheta}{2}$.

9. $$\tan\left(45^\circ \pm \frac{\vartheta}{2}\right) = \frac{\sqrt{1+\cos\vartheta} \pm \sqrt{1-\cos\vartheta}}{\sqrt{1+\cos\vartheta} \mp \sqrt{1-\cos\vartheta}}$$

$$= \frac{1 \pm \sin\vartheta}{\cos\vartheta} = \frac{\cos\vartheta}{1 \mp \sin\vartheta}.$$

10. $$\sin^2\left(45^\circ \pm \frac{\vartheta}{2}\right) = \tfrac{1}{2}(1 \pm \sin\vartheta).$$

11. $$\cos^2\left(45^\circ \pm \frac{\vartheta}{2}\right) = \tfrac{1}{2}(1 \mp \sin\vartheta).$$

12. $$\tan 45^\circ \pm \frac{\vartheta}{2} = \sqrt{\frac{1 \pm \sin\vartheta}{1 \mp \sin\vartheta}}.$$

66. *Functions of Multiple Angles in General.* Put $\beta = (n-1)\,\alpha$ in (32), (33), (34) and (35), using the upper sign; then

$$\alpha + \beta = n\alpha,$$

and

$$\sin n\alpha = \sin\alpha\cos(n-1)\,\alpha + \cos\alpha\sin(n-1)\,\alpha. \quad (52)$$

$$\cos n\alpha = \cos\alpha\cos(n-1)\,\alpha - \sin\alpha\sin(n-1)\,\alpha. \quad (53)$$

$$\tan n\alpha = \frac{\tan\alpha + \tan(n-1)\,\alpha}{1 - \tan\alpha\tan(n-1)\,\alpha}. \quad (54)$$

$$\cot n\alpha = \frac{\cot\alpha\cot(n-1)\,\alpha - 1}{\cot(n-1)\,\alpha + \cot\alpha}. \quad (55)$$

By making $n = 2$, $n = 3$, etc., in succession, expressions for these functions of 2α, 3α, etc., may be successively found in terms of those of α. Thus—

$$\sin 2\alpha = 2\sin\alpha\cos\alpha.$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1.$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$$

$$\cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha}.$$

$$\begin{aligned}\sin 3\alpha &= \sin \alpha \cos 2\alpha + \cos \alpha \sin 2\alpha \\ &= 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha = 3 \sin \alpha - 4 \sin^3 \alpha.\end{aligned}$$

$$\begin{aligned}\cos 3\alpha &= \cos \alpha \cos 2\alpha - \sin \alpha \sin 2\alpha \\ &= \cos^3 \alpha - 3 \sin^2 \alpha \cos \alpha = 4 \cos^3 \alpha - 3 \cos \alpha.\end{aligned}$$

$$\begin{aligned}\tan 3\alpha &= \frac{\tan \alpha + \tan 2\alpha}{1 - \tan \alpha \tan 2\alpha} \\ &= \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}.\end{aligned}$$

$$\cot 3\alpha = \frac{\cot \alpha \cot 2\alpha - 1}{\cot 2\alpha + \cot \alpha} = \frac{\cot^3 \alpha - 3 \cot \alpha}{3 \cot^2 \alpha - 1}.$$

$$\begin{aligned}\sin 4\alpha &= \sin \alpha \cos 3\alpha + \cos \alpha \sin 3\alpha \\ &= 4 \sin \alpha \cos^3 \alpha - 4 \sin^3 \alpha \cos \alpha \\ &= 4 \sin \alpha \cos \alpha - 8 \sin^3 \alpha \cos \alpha.\end{aligned}$$

$$\begin{aligned}\cos 4\alpha &= \cos \alpha \cos 3\alpha - \sin \alpha \sin 3\, \alpha \\ &= \cos^4 \alpha - 6 \sin^2 \alpha \cos^2 \alpha + \sin^4 \alpha \\ &= 8 \cos^4 \alpha - 8 \cos^2 \alpha + 1.\end{aligned}$$

$$\begin{aligned}\tan 4\alpha &= \frac{\tan \alpha + \tan 3\alpha}{1 - \tan \alpha \tan 3\alpha} \\ &= \frac{4 \tan \alpha - 4 \tan^3 \alpha}{1 - 6 \tan^2 \alpha + \tan^4 \alpha}.\end{aligned}$$

$$\begin{aligned}\cot 4\alpha &= \frac{\cot \alpha \cot 3\alpha - 1}{\cot 3\alpha + \cot \alpha} \\ &= \frac{\cot^4 \alpha - 6 \cot^2 \alpha + 1}{4 \cot^3 \alpha - 4 \cot \alpha}.\end{aligned}$$

Exercises.

Given.	*Required.*
$\sin 5^\circ = 0.087$, $\cos 5^\circ = 0.996$,	$\sin 15^\circ$, $\sin 20^\circ$. $\cos 15^\circ$, $\cos 20^\circ$.
$\tan 5^\circ = 0.087$, $\cot 5^\circ = 11.430$,	$\tan 15^\circ$, $\tan 20^\circ$. $\cot 15^\circ$, $\cot 20^\circ$

67. *Sum and Difference of Like Functions of Two Angles.* From (32) and (33), by addition and subtraction, we have

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta. \quad \ldots \quad (56)$$

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta. \quad \ldots \quad (57)$$

$$\cos(\alpha-\beta)+\cos(\alpha+\beta)=2\cos\alpha\cos\beta. \quad \ldots \quad (58)$$

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin\alpha\sin\beta. \quad \ldots \quad (59)$$

Put $\vartheta=\alpha+\beta$, and $\phi=(\alpha-\beta)$; then $\alpha=\frac{1}{2}(\vartheta+\phi)$, and $\beta=\frac{1}{2}(\vartheta-\phi)$, giving

$$\sin\vartheta+\sin\phi=2\sin\tfrac{1}{2}(\vartheta+\phi)\cos\tfrac{1}{2}(\vartheta-\phi). \quad \ldots \quad (60)$$

$$\sin\vartheta-\sin\phi=2\cos\tfrac{1}{2}(\vartheta+\phi)\sin\tfrac{1}{2}(\vartheta-\phi). \quad \ldots \quad (61)$$

$$\cos\phi+\cos\vartheta=2\cos\tfrac{1}{2}(\vartheta+\phi)\cos\tfrac{1}{2}(\vartheta-\phi). \quad \ldots \quad (62)$$

$$\cos\phi-\cos\vartheta=2\sin\tfrac{1}{2}(\vartheta+\phi)\sin\tfrac{1}{2}(\vartheta-\phi). \quad \ldots \quad (63)$$

In (32) put ϑ and ϕ for α and β, and interchange the members, giving

$$\sin\vartheta\cos\phi\pm\cos\vartheta\sin\phi=\sin(\vartheta\pm\phi). \quad \ldots \quad (a)$$

Divide by $\cos\phi\cos\vartheta$, and reduce by (4), giving

$$\tan\vartheta\pm\tan\phi=\frac{\sin(\vartheta\pm\phi)}{\cos\vartheta\cos\phi}. \quad \ldots \quad (64)$$

Divide both members of (a) by $\sin\vartheta\sin\phi$, and reduce by (5), giving

$$\cot\phi \pm \cot\vartheta = \frac{\sin(\vartheta \pm \phi)}{\sin\vartheta\sin\phi}. \quad . \quad . \quad . \quad . \qquad (65)$$

In these equations the operations indicated in the second member are convenient for logarithmic computation. Similar equations may be found for versed-sine, coversed-sine, secant and cosecant.

Let the student verify the following relations:

1. $\text{vers}\,\vartheta - \text{vers}\,\phi = 2\sin\frac{1}{2}(\vartheta + \phi)\sin\frac{1}{2}(\vartheta - \phi)$.

2. $\text{covers}\,\phi - \text{covers}\,\vartheta = 2\cos\frac{1}{2}(\vartheta + \phi)\sin\frac{1}{2}(\vartheta - \phi)$.

3. $\dfrac{\tan\vartheta + \tan\phi}{\tan\vartheta - \tan\phi} = \dfrac{\sin(\vartheta + \phi)}{\sin(\vartheta - \phi)}$.

4. $\sin^2\vartheta - \sin^2\phi = (\sin\vartheta + \sin\phi)(\sin\vartheta - \sin\phi)$
   $= \sin(\vartheta + \phi)\sin(\vartheta - \phi)$.

5. $\tan^2\vartheta - \tan^2\phi = \dfrac{\sin(\vartheta + \phi)\sin(\vartheta - \phi)}{\cos^2\vartheta\cos^2\phi}$.

6. $\cot^2\phi - \cot^2\vartheta = \dfrac{\sin(\vartheta + \phi)\sin(\vartheta - \phi)}{\sin^2\vartheta\sin^2\phi}$.

68. From (60) and (61), by division, we have

$$\frac{\sin\vartheta + \sin\phi}{\sin\vartheta - \sin\phi} = \frac{2\sin\frac{1}{2}(\vartheta + \phi)\cos\frac{1}{2}(\vartheta - \phi)}{2\cos\frac{1}{2}(\vartheta + \phi)\sin\frac{1}{2}(\vartheta - \phi)} = \frac{\tan\frac{1}{2}(\vartheta + \phi)}{\tan\frac{1}{2}(\vartheta - \phi)}. \qquad (66)$$

That is—

The sum of the sines of any two angles is to the difference of their sines as the tangent of half the sum of the angles is to the tangent of half their difference.

Similarly from (60), (61), (62), (63) and (36), after replacing **2α** by $(\vartheta + \phi)$ or $(\vartheta - \phi)$, we have

$$\frac{\sin\vartheta + \sin\phi}{\cos\vartheta + \cos\phi} = \frac{2\sin\frac{1}{2}(\vartheta + \phi)\cos\frac{1}{2}(\vartheta - \phi)}{2\cos\frac{1}{2}(\vartheta + \phi)\cos\frac{1}{2}(\vartheta - \phi)}$$
$$= \tan\tfrac{1}{2}(\vartheta + \phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

$$\frac{\sin\vartheta - \sin\phi}{\cos\vartheta + \cos\phi} = \frac{2\cos\frac{1}{2}(\vartheta + \phi)\sin\frac{1}{2}(\vartheta - \phi)}{2\cos\frac{1}{2}(\vartheta + \phi)\cos\frac{1}{2}(\vartheta - \phi)}$$
$$= \tan\tfrac{1}{2}(\vartheta - \phi). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

$$\frac{\cos\phi - \cos\vartheta}{\cos\phi + \cos\vartheta} = \frac{2\sin\frac{1}{2}(\vartheta + \phi)\sin\frac{1}{2}(\vartheta - \phi)}{2\cos\frac{1}{2}(\vartheta + \phi)\cos\frac{1}{2}(\vartheta - \phi)}$$
$$= \tan\tfrac{1}{2}(\vartheta + \phi)\tan\tfrac{1}{2}(\vartheta - \phi). \quad . \quad . \quad . \quad (69)$$

$$\frac{\sin\vartheta + \sin\phi}{\sin(\vartheta + \phi)} = \frac{2\sin\frac{1}{2}(\vartheta + \phi)\cos\frac{1}{2}(\vartheta - \phi)}{2\sin\frac{1}{2}(\vartheta + \phi)\cos\frac{1}{2}(\vartheta + \phi)}$$
$$= \frac{\cos\frac{1}{2}(\vartheta - \phi)}{\cos\frac{1}{2}(\vartheta + \phi)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

$$\frac{\sin\vartheta - \sin\phi}{\sin(\vartheta - \phi)} = \frac{2\cos\frac{1}{2}(\vartheta + \phi)\sin\frac{1}{2}(\vartheta - \phi)}{2\sin\frac{1}{2}(\vartheta - \phi)\cos\frac{1}{2}(\vartheta - \phi)}$$
$$= \frac{\cos\frac{1}{2}(\vartheta + \phi)}{\cos\frac{1}{2}(\vartheta - \phi)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Comparing (70) and (71), we have

$$\frac{\sin\vartheta + \sin\phi}{\sin(\vartheta + \phi)} = \frac{\sin(\vartheta - \phi)}{\sin\vartheta - \sin\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

That is: *The sum of the sines of any two angles is to the sine of their sum as the sine of their difference is to the difference of their sines.*

CHAPTER V.*

TRIGONOMETRIC DEVELOPMENTS.

69. *To develop* sin x, *and* cos x *into a series arranged according to the ascending powers of* x.

Since $\sin(-x) = -\sin x$, every term in the development of $\sin x$ must change its sign with x, and the development can contain only odd powers of x.

Since $\cos(-x) = \cos x$, no term in the development of $\cos x$ can change its sign with x, and the development can contain only even powers of x.

Let

$$\sin x = A_1 x + A_3 x^3 + A_5 x^5 + A_7 x^7 + \text{etc.}; \quad . \quad . \quad (a)$$
$$\cos x = A_0 + A_2 x^2 + A_4 x^4 + A_6 x^6 + \text{etc.}; \quad . \quad . \quad . \quad (b)$$

in which x is the radian measure of any angle, and A_0, A_1, A_2, A_3, etc., are independent of x and are to be determined.

To find A_1, divide both members of (a) bv x, giving

$$\frac{\sin x}{x} = A_1 + A_3 x^2 + A_5 x^4 + \text{etc.};$$

in which, as x approaches 0, the first member approaches unity as a limit, and the second member A_1. Hence

$$A_1 = 1.$$

To find A_0, make $x = 0$ in (b), giving

$$A_0 = 1.$$

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* Chapters V. and VI. may be omitted by students desiring a shorter course.

To find the remaining coefficients, substitute for A_1 and A_0 their values, and put $(x+y)$ for x in (a) and (b), giving

$$\sin(x+y) = (x+y) + A_3(x+y)^3 + A_5(x+y)^5 + \text{etc.};$$
$$\cos(x+y) = 1 + A_2(x+y)^2 + A_4(x+y)^4 + \text{etc.};$$

or, arranging with reference to y,

$$\sin(x+y) = \left\{\begin{array}{l} x + A_3x^3 + A_5x^5 + \text{etc.} \\ + y(1 + 3A_3x^2 + 5A_5x^4 + 7A_7x^6 + \text{etc.}) \\ + y^2(3A_3x + \text{etc.}) + \text{etc.} \end{array}\right\} \quad . \quad . \quad (c)$$

$$\cos(x+y) = \left\{\begin{array}{l} 1 + A_2x^2 + A_4x^4 + A_6x^6 + \text{etc.} \\ + y(2A_2x + 4A_4x^3 + 6A_6x^5 + \text{etc.}) \\ + y^2\ (\text{etc.}) + \text{etc.} \end{array}\right\} \quad . \quad . \quad (d)$$

But by (32) and (33), we have

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y;$$

or, developing the factors by (a) and (b) with $A_1 = 1$, and $A_0 = 1$, we have

$$\begin{aligned} \sin(x+y) = {} & (x + A_3x^3 + A_5x^5 + \text{etc.})\ (1 + A_2y^2 + \text{etc.}) \\ & + (1 + A_2x^2 + A_4x^4 + \text{etc.})(y + A_3y^3 + \text{etc.}); \end{aligned}$$

$$\begin{aligned} \cos(x+y) = {} & (1 + A_2x^2 + A_4x^4 + \text{etc.})\ (1 + A_2y^2 + \text{etc.}) \\ & - (x + A_3x^3 + A_5x^5 + \text{etc.})(y + A_3y^3 + \text{etc.}); \end{aligned}$$

or, arranging with reference to y,

$$\sin(x+y) = \left\{\begin{array}{l} x + A_3x^3 + A_5x^5 + \text{etc.} \\ + y(1 + A_2x^2 + A_4x^4 + A_6x^6 + \text{etc.}) \\ + A_2y^2\ (\text{etc.}) + \text{etc.} \end{array}\right\} . \quad . \quad . \quad (e)$$

$$\cos(x+y) = \left\{\begin{array}{l} 1 + A_2x^2 + A_4x^4 + \text{etc.} \\ - y(x + A_3x^3 + A_5x^5 + A_7x^7 + \text{etc.}) \\ + A_2y^2\ (\text{etc.}) + \text{etc} \end{array}\right\} . \quad . \quad . \quad (f)$$

Comparing the second members of (c) and (e), we have, by equating the coefficients of y,

$$1 + 3A_3x^2 + 5A_5x^4 + 7A_7x^6 + \text{etc.} = 1 + A_2x^2 + A_4x^4 + A_6x^6 + \text{etc.} \quad (g)$$

In like manner, from (d) and (f), we have

$$2A_2x + 4A_4x^3 + 6A_6x^5 + \text{etc.} = -x - A_3x^3 - A_5x^5 - \text{etc.} \quad (h)$$

By the principle of indeterminate coefficients we have, from (g),

$$3A_3 = A_2, \qquad 5A_5 = A_4, \qquad 7A_7 = A_6, \quad \text{etc.};$$

and from (h),

$$2A_2 = -1, \qquad 4A_4 = -A_3, \qquad 6A_6 = -A_5, \quad \text{etc.}$$

Hence we have, in succession,

$$A_2 = -\frac{1}{2}, \qquad A_3 = \frac{A_2}{3} = -\frac{1}{2.3} = -\frac{1}{|\underline{3}};$$

$$A_4 = -\frac{A_3}{4} = \frac{1}{|\underline{4}}, \qquad A_5 = \frac{A_4}{5} = \frac{1}{|\underline{5}};$$

$$A_6 = -\frac{A_5}{6} = -\frac{1}{|\underline{6}}, \qquad A_7 = \frac{A_6}{7} = -\frac{1}{|\underline{7}};$$

$$\text{etc.,} \qquad \text{etc.}$$

These values, with those of A_1 and A_0 in (a) and (b), give

$$\sin x = x - \frac{x^3}{|\underline{3}} + \frac{x^5}{|\underline{5}} - \frac{x^7}{|\underline{7}} + \text{etc.} \quad . \quad . \quad . \quad . \quad (73)$$

$$\cos x = 1 - \frac{x^2}{|\underline{2}} + \frac{x^4}{|\underline{4}} - \frac{x^6}{|\underline{6}} + \text{etc.}; \quad . \quad . \quad (74)$$

in which x is *expressed in radians.*

To show that (73) and (74) are always convergent, divide the ($n + 1$)th term of (73) by the nth term, giving

$$\frac{(-1)^n x^{2n+1}}{|\underline{2n+1}} \div \frac{(-1)^{n-1} x^{2n-1}}{|\underline{2n-1}} = \frac{-x^2}{2n(2n+1)};$$

which diminishes as n increases.

Take n' a particular value for n large enough to make $\frac{-x^2}{2n'(2n'+1)}$ numerically less than unity.

In the geometrical progression

$$a + av + av^2 + av^3 + \text{etc.}, \quad . \quad . \quad . \qquad . \qquad (k)$$

put $a = n$th term of (73), and $v = \frac{-x^2}{2n'(2n'+1)}$.

Then $av = (n+1)$th term of (73); and since $\frac{-x^2}{2n(2n+1)}$ continually diminishes as n increases, while v remains constant, the $(n+2)$th, $(n+3)$rd, etc., terms of (73) are each numerically less than av^2, av^3, etc., respectively, and have the same algebraic sign.

Hence, the sum of the terms of (73), after the $(n-1)$th term, is numerically less than the sum of the terms of (k), which is convergent since v is numerically less than unity.

Hence, (73) is convergent for all values of x.

In like manner, it may be shown that (74) is also convergent.

(73) and (74) enable us to compute the values of sin x and cos x to any required degree of accuracy.

Thus, to compute sin 1°, we have

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \text{etc.}$$

$$x = r \text{ measure } 1^\circ = \frac{\pi}{180}.$$

$$x = 0.01745329$$

$$x^2 = 0.00030462$$

$$\frac{x^3}{6} = 0.000000886$$

$$\frac{x^5}{6} = 0.00000000027$$

$$\frac{x^5}{\underline{|5}} = 0.00000000001$$

$$\begin{array}{rl}
+ x = & + 0.01745329 \\
- \frac{x^3}{6} = & 0.000000886 \\
+ \frac{x^5}{\underline{|5}} = & 0.00000000001 \\
\hline
\sin 1^\circ = & 0.01745240
\end{array}$$

Examples.

Compute, by (73) and (74), accurately to 6 decimal places:

1. $\sin 5^\circ$. *Ans.* 0.087156.
2. $\cos 10^\circ$. *Ans.* 0.984808.
3. $\sin 15^\circ$. *Ans.* 0.258819.
4. $\sin 30^\circ$. *Ans.* 0.500000.

70. *Development of powers, and products of powers, of the sine and cosine of any angle, in terms of the sine and cosine of integral multiples of the angle.*

In (56), (57), (58) and (59), interchanging the members, we have

$$\begin{aligned}
2 \sin \alpha \cos \beta &= \sin (\alpha + \beta) + \sin (\alpha - \beta);\\
2 \cos \alpha \sin \beta &= \sin (\alpha + \beta) - \sin (\alpha - \beta);\\
2 \cos \alpha \cos \beta &= \cos (\alpha - \beta) + \cos (\alpha + \beta);\\
2 \sin \alpha \sin \beta &= \cos (\alpha - \beta) - \cos (\alpha + \beta);
\end{aligned}$$

in which put $\alpha = nx$, and $\beta = x$, giving

$$\left.\begin{aligned}
2 \sin nx \cos x &= \sin (n + 1)x + \sin (n - 1)x;\\
2 \cos nx \sin x &= \sin (n + 1)x - \sin (n - 1)x;\\
2 \cos nx \cos x &= \cos (n - 1)x + \cos (n + 1)x;\\
2 \sin nx \sin x &= \cos (n - 1)x - \cos (n + 1)x.
\end{aligned}\right\} . \qquad (a)$$

Putting $n = 1$, we have, from 4th, 1st and 3d of (a), respectively,

$$\left.\begin{aligned}
2 \sin^2 x &= 1 - \cos 2x;\\
2 \sin x \cos x &= \sin 2x;\\
2 \cos^2 x &= \cos 2x + 1.
\end{aligned}\right\} \text{[Compare (38), (36) and (39)].}$$

Multiply each of these by $2 \sin x$, and the last by $2 \cos x$, giving

$$\left.\begin{aligned}
2^2 \sin^3 x &= 2 \sin x - 2 \cos 2x \sin x;\\
2^2 \sin^2 x \cos x &= 2 \sin 2x \sin x;\\
2^2 \sin x \cos^2 x &= 2 \cos 2x \sin x + 2 \sin x;\\
2^2 \cos^3 x &= 2 \cos 2x \cos x + 2 \cos x.
\end{aligned}\right\} \qquad (74^*)$$

Reduce by (a), when $n = 2$, giving

$$\left.\begin{aligned} 2^2 \sin^3 x &= 3 \sin x - \sin 3x; \\ 2^2 \sin^2 x \cos x &= \cos x - \cos 3x; \\ 2^2 \sin x \cos^2 x &= \sin 3x + \sin x; \\ 2^2 \cos^3 x &= \cos 3x + 3 \cos x \end{aligned}\right\} . \quad . \;\; . \;\; . \quad (75)$$

Multiply each of these by $2 \sin x$, and the last by $2 \cos x$, giving

$$\begin{aligned} 2^3 \sin^4 x &= 6 \sin^2 x - 2 \sin 3x \sin x; \\ 2^3 \sin^3 x \cos x &= 2 \cos x \sin x - 2 \cos 3x \sin x; \\ 2^3 \sin^2 x \cos^2 x &= 2 \sin 3x \sin x + 2 \sin^2 x; \\ 2^3 \sin x \cos^3 x &= 2 \cos 3x \sin x + 6 \cos x \sin x; \\ 2^3 \cos^4 x &= 2 \cos 3x \cos x + 6 \cos^2 x. \end{aligned}$$

Reduce by (a), when $n = 1$, and $n = 3$, giving

$$\left.\begin{aligned} 2^3 \sin^4 x &= 3 - 4 \cos 2x + \cos 4x; \\ 2^3 \sin^3 x \cos x &= 2 \sin 2x - \sin 4x; \\ 2^3 \sin^2 x \cos^2 x &= 1 - \cos 4x; \\ 2^2 \sin x \cos^3 x &= \sin 4x + 2 \sin 2x; \\ 2^3 \cos^4 x &= \cos 4x + 4 \cos 2x + 3 \end{aligned}\right\} . \;\; . \;\; . \quad (76)$$

In like manner, by multiplication and reduction, we have

$$\left.\begin{aligned} 2^4 \sin^5 x &= 10 \sin x - 5 \sin 3x + \sin 5x; \\ 2^4 \sin^4 x \cos x &= 2 \cos x - 3 \cos 3x + \cos 5x; \\ 2^4 \sin^3 x \cos^2 x &= 2 \sin x + \sin 3x - \sin 5x; \\ 2^4 \sin^2 x \cos^3 x &= 2 \cos x - \cos 3x - \cos 5x; \\ 2^4 \sin x \cos^4 x &= \sin 5x + 3 \sin 3x + 2 \sin x; \\ 2^4 \cos^5 x &= \cos 5x + 5 \cos 3x + 10 \cos x. \end{aligned}\right\} \quad (77)$$

This process may be continued to obtain expressions for powers and products of any required degree. It is only necessary to divide by the numerical coefficient in the first member in order to obtain the required expression.

71. *Law of Development for powers of* $\cos x$. By inspecting the last equation of each of the groups (74*), (75), (76) and (77), we see: that the coefficient of the power in the first member is a power of 2, with an exponent less by unity than that of cos x; that the functions in the second member are all cosines; that the multiple angles begin with the multiplier equal to the exponent of cos x; that the multiplier diminishes by 2 in each succeeding term until it becomes unity or zero; that the coefficients are all positive, and follow the binomial law for the power considered, except the term independent of the angle; that this term enters only for the even powers, and is equal to half of the corresponding binomial coefficient.

This law may be proved for all integral powers of cos x, by assuming it true for the $(2n - 1)$ power, and from it, by multiplication and reduction, forming the $2n$th and $(2n + 1)$th power. The law will be found to be true for these powers also. But, by (77), it is true when $(2n - 1) = 5$, thence for the 6th and 7th powers; thence for the 8th and 9th powers, etc., indefinitely.

By the law,

$$*2^{n-1}\cos^n x = \cos nx + \left(\frac{n}{1}\right)\cos(n-2)x + \left(\frac{n}{2}\right)\cos(n-4)x + \text{etc.} \quad (78)$$

The binomial law applies throughout if n is odd, but if n is even the last term is $\frac{1}{2}\left(\frac{n}{\frac{1}{2}n}\right)$, or, when written in full,

$$\frac{1}{2}\cdot\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{(n-\frac{1}{2}n+1)}{\frac{1}{2}n}.$$

---

* In the abbreviated binomial notation,

$\left(\frac{n}{1}\right)$ denotes $\frac{n}{1} = n.$

$\left(\frac{n}{2}\right)$ denotes $\frac{n}{1}\cdot\frac{n-1}{2} = \frac{n(n-1)}{\underline{|2}}.$

$\left(\frac{n}{3}\right)$ denotes $\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} = \frac{n(n-1)(n-2)}{\underline{|3}}.$

etc. etc. etc.

$\left(\frac{n}{r}\right)$ denotes $\frac{n}{1}\cdot\frac{n-1}{2}\cdots\frac{n-(r-1)}{r} = \frac{n(n-1)\ldots(n-[r-1])}{\underline{|r}}.$

etc. etc. etc.

72. Any power of $\sin x$ may be developed by putting $90° - x$ for x in the development of the same power of $\cos x$.

Thus, to find $\sin^6 x$ and $\sin^7 x$,

$$2^5 \cos^6 x = \cos 6x + 6 \cos 4x + 15 \cos 2x + 10.$$

Hence,

$$2^5 \sin^6 x = \cos (540° - 6x) + 6 \cos (360° - 4x) + 15 \cos (180° - 2x) + 10,$$

or

$$2^5 \sin^6 x = - \cos 6x + 6 \cos 4x - 15 \cos 2x + 10.$$

Similarly,

$$2^6 \cos^7 x = \cos 7x + 7 \cos 5x + 21 \cos 3x + 35 \cos x;$$
$$2^6 \sin^7 x = - \sin 7x + 7 \sin 5x - 21 \sin 3x + 35 \sin x.$$

Examples.

Reduce the following expressions to others consisting of the sum and difference of sines and cosines of integral multiples of x:

1. $\sin^{10} x + \cos^{10} x$. *Ans.* $\frac{1}{128}\{5 \cos 8x + 60 \cos 4x + 63\}$.

2. $8 \sin^5 x + 8 \cos^5 x$.
Ans. $\frac{1}{2}\{\cos 5x + \sin 5x + 5 \cos 3x - 5 \sin 3x + 10 \cos x + 10 \sin x\}$.

3. $32 \sin^6 x + 96 \cos^2 x \sin^2 x + 32 \cos^6 x - 30$. *Ans.* 2.

4. $16 \cos^5 x - 20 \cos^3 x + 5 \cos x$. *Ans.* $\cos 5x$.

5. $5 \cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x$. *Ans.* $\sin 5x$.

6. $8 \cos^3 x \sin x - 4 \cos x \sin x + 8 \cos^4 x - 8 \cos^2 x + 1$.
Ans. $\sin 4x + \cos 4x$.

NOTE.—The construction of trigonometric tables, and the methods of using them in trigonometric computations, should now be studied.

CHAPTER VI.

SOLUTION OF TRIGONOMETRIC EQUATIONS.

73. *Literal Equations.* If any function of a required angle be exactly expressed in terms of given quantities, the angle may be expressed as an *inverse function* (§ 50). Thus, if $\sin\phi = a$, we have $\phi = \sin^{-1} a$. If each required angle enters the equations by its direct functions only, we may, by aid of the formulas of Table B, reduce different functions of the same angle to expressions involving a single function of that angle, which function may then be treated as an algebraic required quantity. If these equations can be solved, each value of this function, found by algebraic methods, will give one or more values of the angle, according to the conditions of the problem under consideration. Thus, let

$$r \sin\phi = a,$$
$$r \cos\phi = b;$$

in which a and b are given, r and ϕ required. By (1) we have

$$\cos\phi = \pm\sqrt{1 - \sin^2\phi};$$

giving, after eliminating r,

$$\frac{a}{\sin\phi} = \frac{b}{\pm\sqrt{1 - \sin^2\phi}},$$

whence $\sin\phi = \pm\frac{a}{\sqrt{a^2 + b^2}}, \quad \cos\phi = \pm\frac{b}{\sqrt{a^2 + b^2}},$

or $$\phi = \sin^{-1}\left(\pm \frac{a}{\sqrt{a^2+b^2}}\right) = \cos^{-1}\left(\pm \frac{b}{\sqrt{a^2+b^2}}\right),$$

and $$r = \pm \sqrt{a^2+b^2}.$$

Examples.

Solve the following equations:

1. $\sin\phi = n\cos\phi$.

$$\textit{Ans.}\ \phi = \sin^{-1}\left(\pm \frac{n}{\sqrt{1+n^2}}\right), \quad \text{or} \quad \cos^{-1}\left(\pm \frac{1}{\sqrt{1+n^2}}\right).$$

2. $\sin\phi = n\cot\phi$.

$$\textit{Ans.}\ \phi = \sin^{-1}\left[\pm \sqrt{\frac{-n^2 \pm n\sqrt{n^2+4}}{2}}\right].$$

3. $m\cos\phi + n\sin\phi = q$.

$$\textit{Ans.}\ \phi = \cos^{-1}\left[\frac{mq \pm n\sqrt{m^2+n^2-q^2}}{m^2+n^2}\right], \text{or} \sin^{-1}\left[\frac{nq \pm m\sqrt{m^2+n^2-q^2}}{m^2+n^2}\right].$$

4. Given $a^2\sin^2\alpha + b^2\cos^2\alpha = a^2\sin^2\alpha' + b^2\cos^2\alpha'$,

show that $$\cos\alpha = \pm\cos\alpha',$$

and from this result show, when the conditions of the problem require α and α' to have different values, each less than 180°, that

$$\alpha + \alpha' = 180^\circ.$$

5. Find the angle whose sine equals its cotangent.

Ans. 51° 49′ 38″.

74. Elimination by the relation $\sin^2\phi + \cos^2\phi = 1$ is often convenient.

To illustrate, let

$$r \sin\phi = a,$$
$$r \cos\phi = b.$$

Square both members of each equation and add, giving

$$r^2 (\sin^2\phi + \cos^2\phi) = a^2 + b^2;$$

whence $$r^2 = a^2 + b^2,$$

or $$r = \pm\sqrt{a^2 + b^2},$$

and $$\phi = \sin^{-1}\left(\frac{a}{r}\right) = \sin^{-1}\left(\pm\frac{a}{\sqrt{a^2 + b^2}}\right),$$
$$\phi = \cos^{-1}\left(\frac{b}{r}\right) = \cos^{-1}\left(\pm\frac{b}{\sqrt{a^2 + b^2}}\right).$$

Examples.

1. Solve the equations

$$r \cos v \cos u = a,$$
$$r \cos v \sin u = b,$$
$$r \sin v \qquad = c.$$

Ans. $r = \pm\sqrt{a^2 + b^2 + c^2},$

$$v = \sin^{-1}\left[\pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right], \quad u = \sin^{-1}\left[\pm\frac{b}{\sqrt{a^2 + b^2}}\right].$$

2. Given $$x \cos\alpha + y \sin\alpha = s,$$
$$x \sin\alpha - y \cos\alpha = t,$$

show that $$x = s \cos\alpha + t \sin\alpha,$$
$$y = s \sin\alpha - t \cos\alpha.$$

Suggestion. Eliminate x or y and simplify.

3. Given
$$x \sin \alpha + y \cos \alpha = p,$$
$$x \cos \alpha - y \sin \alpha = q,$$
show that
$$x = p \sin \alpha + q \cos \alpha,$$
$$y = p \cos \alpha - q \sin \alpha.$$

4. Given
$$r^2 \sin^2 v + 2pr \cos v - p^2 = 0,$$
show that
$$r = p\frac{\cos v \pm 1}{\sin^2 v} = \frac{p}{1 \mp \cos v}$$

75. Elimination by division is often used. Thus, having
$$r \sin \phi = a,$$
$$r \cos \phi = b.$$

By division, since $\frac{\sin \phi}{\cos \phi} = \tan \phi$, we have
$$\tan \phi = \frac{a}{b},$$
or
$$\phi = \tan^{-1}\left(\frac{a}{b}\right).$$

Since
$$\sin \phi = \pm \frac{\tan \phi}{\sqrt{1 + \tan^2 \phi}} = \pm \frac{a}{\sqrt{a^2 + b^2}},$$
we have
$$r = \pm \sqrt{a^2 + b^2}$$

Examples.

1. Given a and b, find α and α' subject to the condition that they shall differ, and each be $< 180°$.
$$a^2 \sin^2 \alpha - b^2 \cos^2 \alpha = 0,$$
$$a^2 \sin^2 \alpha' - b^2 \cos^2 \alpha = 0.$$
$$\textit{Ans.}\ \alpha = \tan^{-1}\left(\pm \frac{b}{a}\right), \quad \alpha' = \tan^{-1}\left(\mp \frac{b}{a}\right).$$

2. Given a and b, find p and α.

$$b^2 = 2pa,$$
$$b \sin \alpha + p \cos \alpha = 0.$$

$$\textit{Ans.}\ p = \frac{b^2}{2a}, \qquad \alpha = \tan^{-1}\left(\frac{-b}{2a}\right)$$

3. Given a, b and c in the equations

$$r \cos v \cos u = a,$$
$$r \cos v \sin u = b,$$
$$r \sin v = c,$$

find u, v and r.

$$\textit{Ans.}\ u = \tan^{-1}\left(\frac{b}{a}\right), \qquad v = \tan^{-1}\left(\pm \frac{c}{\sqrt{a^2 + b^2}}\right),$$

$$r = \pm \sqrt{a^2 + b^2 + c^2}.$$

4. From the equations

$$4a^2 = (b^2 - 4ac) \cos^2 \alpha,$$

$$\tan \alpha = - \frac{b}{2a},$$

show that $$a = -c.$$

Suggestion. Eliminate α, b disappears.

5. From the equations

$$y = Vt \sin \phi - \tfrac{1}{2} gt^2,$$

$$x = Vt \cos \phi,$$

show that $$V^2t^2 = x^2 + (y + \tfrac{1}{2} gt^2)^2.$$

$$y = x \tan \phi - \frac{g}{2} \cdot \frac{x^2}{V^2 \cos^2 \phi}.$$

76. The formulas for functions of the sum or difference of two angles may sometimes be used to put an equation under the forms considered above. Thus to solve

$$\sin(\alpha \pm \phi) = m \sin \phi,$$

α and m being given, ϕ required, we have, by (32),

$$\sin(\alpha \pm \phi) = \sin \alpha \cos \phi \pm \cos \alpha \sin \phi;$$

giving

$$\sin \alpha \cos \phi \pm \cos \alpha \sin \phi = m \sin \phi.$$

Divide both members by $\sin \alpha \sin \phi$, giving

$$\cot \phi \pm \cot \alpha = \frac{m}{\sin \alpha};$$

$$\therefore \quad \cot \phi = m \operatorname{cosec} \alpha \mp \cot \alpha,$$

$$\phi = \cot^{-1}[m \operatorname{cosec} \alpha \mp \cot \alpha].$$

Examples.

1. Find ϕ in the equation

$$\sin(\alpha + \phi) = p \cos \phi.$$

Ans. $\phi = \tan^{-1}(p \sec \alpha - \tan \alpha)$.

2. Find ϕ in the equation

$$\cos(\alpha - \phi) = n \cos \phi.$$

Ans. $\phi = \tan^{-1}(n \operatorname{cosec} \alpha - \cot \alpha)$.

3. Find ϑ in the equation

$$\tan(\alpha + \vartheta) = m \cot \vartheta.$$

$$Ans.\ \vartheta = \tan^{-1}\left[\frac{\tan \alpha}{m-1} \pm \sqrt{m + \left(\frac{\tan \alpha}{m-1}\right)}\right].$$

4. Given $\tan(\alpha+\phi)=k$, find ϕ.

Ans. $\phi=-\alpha+\tan^{-1}(k)$.

5. Given

$$a+r\cos(v+\alpha)=k,$$
$$b+r\sin(v+\alpha)=l,$$

find r and v.

Ans. $r=\pm\sqrt{(k-a)^2+(l-b)^2}$. $v=-\alpha+\cos^{-1}\left[\pm\frac{k-a}{\sqrt{(k-a)^2+(l-b)^2}}\right]$

77. The formulas for functions of integral multiples of angles may be used to reduce a complex equation to one of the simple forms already treated (§ 73).

Examples.

1. Given $\cos 2\phi=n\sin^2\phi$, find ϕ.

Ans. $\phi=\cot^{-1}(\pm\sqrt{n+1})$

2. Given $\sin 2\phi=n\sin^2\phi$, find ϕ *Ans.* $\phi=\cot^{-1}\left(\frac{n}{2}\right)$

3. Given $\cos 2\vartheta=2m\cos\vartheta$, find ϑ.

Ans. $\cos^{-1}\left[\frac{m\pm\sqrt{2+m^2}}{2}\right]$

4. Given

$$\tan\alpha\tan\alpha'=-k^2,$$
$$\alpha+\alpha'=180^\circ,$$

find α and α'. *Ans.* $\alpha=\tan^{-1}[\pm k]$, $\alpha'=\tan^{-1}[\mp k]$.

5. Given

$$\tan\alpha\tan\alpha'=-\frac{b^2}{a^2},$$
$$a'b'\sin(\alpha'-\alpha)=ab,$$
$$a'^2+b'^2=a^2+b^2,$$
$$\alpha'+\alpha=180^\circ.$$

find α and α', a' and b' in terms of a and b, subject to the condition that α and α' are each $< 180°$ and positive.

$$\textit{Ans.}\ \alpha = \tan^{-1}\left(\pm \frac{b}{a}\right), \qquad \alpha' = \tan^{-1}\left(\mp \frac{b}{a}\right),$$

$$a' = \pm\sqrt{\frac{a^2+b^2}{2}}, \qquad b' = \pm\sqrt{\frac{a^2+b^2}{2}},$$

$$\sin(\alpha'-\alpha) = \left(+\frac{b}{\sqrt{a^2+b^2}}\right)\left(\pm\frac{a}{\sqrt{a^2+b^2}}\right) - \left(\mp\frac{a}{\sqrt{a^2+b^2}}\right)\left(+\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$= \pm\frac{2ab}{\sqrt{a^2+b^2}}.$$

Numerical Equations.

78. If any function of a required angle is given numerically, we may express it by using the inverse function, or we may at once compute the angle by aid of the trigonometric tables.

The given value, reduced to a decimal, is used with the table of natural functions, and its logarithm, increased by 10 if necessary, with the table of logarithmic functions. The latter is usually preferred.

Thus, given $\tan\phi = -\frac{2}{3}$, to find ϕ from the table of common logarithms.

$$\log 2 = 0.30103$$
$$\log 3 = 0.47712$$

$$\log\tan\phi = [\log 2 - \log 3]_n = \bar{1}.82391_n$$

$$\log\tan\phi + 10 = 9.82391_n;$$

$$\therefore \begin{cases} \phi = 180° - (33°\ 41'\ 24'') = 146°\ 18'\ 36'', \\ \phi = 360° - (33°\ 41'\ 24'') = 326°\ 18'\ 36''. \end{cases}$$

Similarly, given $\cot\phi = \dfrac{\cos\ 30°\ 45'\ 10''\ \tan\ 190°\ 15'\ 20''}{\sec\ 150°\ 25'\ 30''\ \sin\ 330°\ 25'\ 30''}$, to find ϕ, we have

$$\begin{aligned}\log\cot\phi = {}& \log\cos\ 30^\circ\ 45'\ 10'' + \log\tan 190^\circ\ 15'\ 20''\\ & - \log\sec 150^\circ\ 25'\ 30''\\ & - \log\sin\ 330^\circ\ 25'\ 30''\end{aligned}$$

$$\begin{array}{rr} + \log\cos\ 30^\circ\ 45'\ 10'' & 9.93419_p - 10\\ + \log\tan 190^\circ\ 15'\ 20'' & 9.25751_p - 10\\ - \log\sec 150^\circ\ 25'\ 30'' = + \log\cos 150^\circ\ 25'\ 30'' & 9.9393\hat{7}_n - 10\\ - \log\sin\ 330^\circ\ 25'\ 30'' = 10 - 9.6933\hat{4}_n \quad & = 0.3066\hat{5}_n\\ \hline & 29.43773_p - 30\\ \log\cot\phi & 9.43773_p - 10 \end{array}$$

$$\therefore\ \phi = 74^\circ\ 40'\ 40'' \qquad \text{and} \qquad \phi = 254^\circ\ 40'\ 40''.$$

Examples.

Compute by the Table of Logarithmic Trigonometric Functions the following expressions:

1. $\sin^{-1}\left(\frac{-2}{5}\right) = \begin{cases} 203^\circ\ 34'\ 41'', \\ 336^\circ\ 25'\ 19''. \end{cases}$

2. $\cot^{-1}\left(\frac{3}{-4}\right) = \begin{cases} 127^\circ\ 52'\ 11'', \\ 307^\circ\ 52'\ 11''. \end{cases}$

3. $\sec^{-1} 3 = \begin{cases} 70^\circ\ 31'\ 43'', \\ 289^\circ\ 28'\ 17''. \end{cases}$

4. $\tan^{-1}(\text{vers } 130^\circ) = \begin{cases} 58^\circ\ 40'\ 13'', \\ 238^\circ\ 40'\ 13''. \end{cases}$

5. $\tan^{-1}\left(\frac{[\cot 215^\circ\ 41'\ 10'']^6}{\sqrt[4]{\cos 152^\circ\ 44'\ 40''}}\right) =$

6. $\cot^{-1}\sqrt[5]{\frac{\cos 162^\circ\ 13'\ 15''}{\tan 157^\circ\ 55'\ 15''}} = \begin{cases} 40^\circ\ 8'\ 4'', \\ 220^\circ\ 8'\ 4''. \end{cases}$

79. The methods already indicated with literal equations for expressing each required angle in terms of the given quantities may be applied to numerical equations, after which each required quantity may be computed.

Examples.

Find in degrees each positive value of ϕ less than $360°$:

1. When $\sin\phi = 2\cos\phi$.

By division. $\tan\varphi = 2;$

$$\therefore \log\tan\phi = \log 2 = 0.30103;$$

$$\therefore \phi = 63° \, 26' \, 07'' \text{ or } 243° \, 26' \, 07''.$$

2. When $\cos 2\phi + \sin\phi = 1$.
Ans. $0°$, $180°$, $30°$, or $150°$.

3. When $4\sec^2\phi - 5\tan^2\phi = 1$.
Ans. $60°$, $120°$, $240°$, $300°$.

80. In many cases it is more convenient to compute some of the required quantities from others computed first. Four important cases will be considered.

CASE I. Given
$$r\sin\phi = a,$$
$$r\cos\phi = b,$$

in which a and b are given algebraic numbers, r and ϕ are required.

By division, as in § 75, we have

$$\tan\phi = \frac{a}{b},$$

from which ϕ may be computed.

We also have, from the given equations,

$$r = \frac{a}{\sin\ \phi} \qquad \text{and} \qquad r = \frac{b}{\cos\ \phi},$$

from which to compute r when ϕ is known.

In order to check the computation of ϕ, r should be computed from each of these formulas separately. The two results should agree within the limit of unavoidable tabular errors. A discrepancy of more than two tabular units usually indicates a mistake. In case of an allowable discrepancy, the mean of the two values of log r is preferable to either.

Examples.

1. Given

$$r \sin\ \phi = 332.76,$$

$$r \cos\ \phi = 290.08,$$

to find r and ϕ, both being positive, and $\phi < 360°$.

$$\tan\ \phi = \frac{332.76}{290.08}, \qquad \text{and} \qquad r = \frac{332.76}{\sin\ \phi} = \frac{290.08}{\cos\ \phi}.$$

$$\begin{array}{rl} \log 332.76 & 2.5221\hat{3}_p \\ \text{a. c.} \log 290.08 & 7.53748_p - 10 \\ \hline \log \tan\ \phi & 0.0596\hat{1} \end{array}$$

$$\therefore \quad \phi = 48° \ 55' \ 13'',$$

the only admissible value since sin ϕ and cos ϕ are both positive.

$$\begin{array}{rl} \log 322.76 & 2.5221\hat{3}_p \\ \text{a. c.} \log \sin\ \phi & 0.1227\hat{4}_p \\ \hline \log r & 2.64488_p \end{array} \qquad \begin{array}{rl} \log 290.88 & 2.46252_p \\ \text{a. c.} \log \cos\ \phi & 0.18236_p \\ \hline \log r & 2.64488_p \end{array}$$

$$\therefore \ r = 441.145.$$

In the following examples find r and ϕ subject to the same limitation as above:

2. Given $r \sin \phi = 1.2974$, $r \cos \phi = 60.0243$.
Ans. $\phi = 1^\circ\ 14'\ 17''.6$; $r = 60.0366$.

3. Given $r \sin \phi = -0.0822$, $r \cos \phi = 0.1288$.
Ans. $\phi = 327^\circ\ 27'\ 15''$, $r = 0.1528$.

4. Given $r \sin \phi = 19468.3$, $r \cos \phi = -846.06$.
Ans. $\phi = 92^\circ\ 29'\ 18''$. $r = 19486.8$.

5. Given $r \sin \phi = -63.824$, $r \cos \phi = -521.335$.
Ans. $\phi = 186^\circ\ 58'\ 46''$, $r = 525.23$.

Case II. Given

$$\begin{aligned} r \cos v \cos u &= a, \\ r \cos v \sin u &= b, \\ r \sin v &= c; \end{aligned}$$

in which a, b and c are given algebraic numbers, and r, u and v are required; r, u and v being positive, $u < 180^\circ$ and $v < 360^\circ$.

From the first and second of these, by division, we have

$$\tan u = \frac{b}{a};$$

from which u may be computed.

In like manner, from the first and third,

$$\tan v = \frac{c \cos u}{a};$$

and from the second and third,

$$\tan v = \frac{c \sin u}{b};$$

from each of which v should be computed. A comparison of the two values checks the computed value of u.

From the given equations,

$$r = \frac{a}{\cos v \cos u},$$

$$r = \frac{b}{\cos v \sin u},$$

$$r = \frac{c}{\sin v}.$$

r should be computed from the last of these, and also from one of the others, to check the value of v.

Examples.

Find r, u and v, all being positive, $u < 180°$ and $v < 360°$

1. When

$$\begin{aligned} r \cos v \cos u &= -\ 53.953, \\ r \cos v \sin u &= 197.207, \\ r \sin v &= -\ 39.062, \end{aligned}$$

$$\begin{aligned} &\log r \cos r \sin u &&= 2.29492_p \\ &\log r \cos v \cos u &&= \underline{1.7320\hat{1}_n} \\ &\text{Difference, } \log \tan u &&= 0.5629\hat{0}_n \\ &\qquad \therefore\ u = 105^\circ\ 18'\ 4'' \\ &\log r \sin v &&= 1.5917\hat{5}_n \\ &\text{a. c. } \log r \cos v \cos u &&= 8.2679\hat{8}_n \\ &+ \log \cos u &&= \underline{9.4214\hat{2}_n} \\ &\log \tan v &&= 9.2811\hat{6}_n \\ &\qquad \therefore\ v = 349^\circ\ 11'\ 1'. \end{aligned}$$

Check.

$$\begin{aligned} &\log r \sin v &&= 1.5917\hat{5}_n \\ &\text{a. c. } \log r \cos v \sin u &&= 7.7050\hat{8}_p \\ &\log \sin u &&= \underline{9.9843\hat{2}_p} \\ &\log \tan v &&= 9.2811\hat{6}_n \\ &\log r \cos v \cos u &&= 1.7320\hat{1}_n \\ &\text{a. c. } \log \cos u &&= 0.5785\hat{7}_n \\ &\text{a. c. } \log \cos v &&= \underline{0.0077\hat{8}} \\ &\therefore\ \log r &&= 2.3183\hat{7}_p \end{aligned}$$

The value $v = 169^\circ\ 11'\ 1''$ is rejected, since $\sin v$ is negative.

$$\begin{aligned} &\log r \sin v &&= 1.5917\hat{5}_n \\ &\log \sin v &&= \underline{9.27338_n} \\ &\text{Difference,}\quad \log r &&= 2.3183\hat{7}_p; \quad \therefore\ r = 208.15 \end{aligned}$$

2. When

$$r \cos v \cos u = 1.27,$$
$$r \cos v \sin u = -0.98,$$
$$r \sin v = 0.89.$$

Ans. $\log \tan u = 9.88742_n$, $u = 142^\circ\ 20'\ 40''$,
$\log \tan v = 9.74414_n$, $v = 150^\circ\ 58'\ 41''$,
$\log r = 0.26352_p$, $r = 1.8345$

3. When

$$r \cos v \cos u = -19.76,$$
$$r \cos v \sin u = -7.19,$$
$$r \sin v = 12.12.$$

Ans. $u = 19^\circ\ 59'\ 41''$, $v = 150^\circ\ 2'\ 29''$, $r = 24.271$.

CASE III. Given

$$\sin(2\alpha \pm \phi) = m \sin \phi,$$

in which α and m are known algebraic numbers.

From this equation, we have

$$\frac{\sin(2\alpha + \phi) + \sin \phi}{\sin(2\alpha + \phi) - \sin \phi} = \frac{m \sin \phi + \sin \phi}{m \sin \phi - \sin \phi} = \frac{m + 1}{m - 1}.$$

From (66),

$$\frac{\sin(2\alpha + \phi) + \sin \phi}{\sin(2\alpha + \phi) - \sin \phi} = \frac{\tan(\alpha + \phi)}{\tan \alpha};$$

$$\therefore \quad \tan(\alpha + \phi) = \frac{m + 1}{m - 1} \tan \alpha.$$

In like manner, it may be shown that

$$\tan(\alpha - \phi) = \frac{m - 1}{m + 1} \tan \alpha;$$

$$\therefore \quad \tan(\alpha \pm \phi) = \frac{m \pm 1}{m \mp 1} \tan \alpha;$$

from which $(\alpha \pm \phi)$ may be computed, and thence ϕ. Compare § 77.

Examples.

1. $\sin(30° + \phi) = 7.93 \sin \phi$.
Ans. $\phi = 4° 2' 56''$ or $184° 2' 56''$.

2. $\sin(12° 15' - \phi) = .00371 \sin \phi$.
Ans. $\phi = 12° 12' 18''$ or $192° 12' 18''$.

3. $\sin(99° 50' + \phi) = -0.015 \sin \phi$.
Ans. $\phi = 81° 0' 56''$ or $261° 0' 56''$.

4. $\sin(20° 20' - \phi) = -3.21 \sin \phi$.
Ans. $\phi = 171° 18' 20''$ or $351° 18' 20''$.

CASE IV. Given $m \cos \phi + n \sin \phi = q$, ... (a)

in which ϕ is required, and m, n, q are given algebraic numbers.

Let $x \sin y = m$, and $x \cos y = n$.

In (a), substitute for m and n their values in terms of x and y, giving

$$x \sin y \cos \phi + x \cos y \sin \phi = q,$$

or, by (28), $$\sin(y + \phi) = \frac{q}{x}. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (b)$$

Eliminating x from (b), we have

$$\sin(y + \phi) = \frac{q \sin y}{m} = \frac{q \cos y}{n}; \quad . \ . \ . \ . \ . \quad (c)$$

and since

$$\tan y = \frac{m}{n},$$

ϕ may be determined.

Double computation should be made, as indicated, to check the value of y, which is auxiliary, and should be limited to a positive value less than 180°.

(*b*) shows that the solution is impossible if q is numerically greater than x, or since $x = \pm \sqrt{m^2 + n^2}$, if

$$q^2 > m^2 + n^2,$$

which also limits the possibility of solving (*c*).

Examples.

Compute ϕ:

1. When $9.863 \cos \phi + 6.821 \sin \phi = -1.799$.
Ans. $y = 55° 20'$, $\phi = 133° 17' 40''.5$ or $347° 2' 19''.5$.

2. When $36.85 \sin \phi - 11.45 \cos \phi = 28.34$.
Ans. $y = 162° 44' 20''$, $\phi = 64° 31' 13''$ or $150° 0' 07''$.

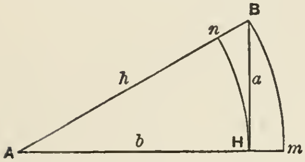
3. When $348.5 \cos \phi - 243.7 \sin \phi = -356.8$.
Ans. $y = 124° 57' 51''$, $\phi = 112° 4' 24''$ or $177° 59' 54''$.

CHAPTER VII.

RIGHT PLANE TRIANGLES.

Relations between Sides, Angles and Areas.

81. Denote the angles of the right plane triangle *AHB* by *A*, $H = \frac{\pi}{2}$ and *B*; and their opposite sides by *a*, *h* and *b*, respectively.



Let Δ = area of *AHB*.

With *A* as a centre, and radius *AB*, describe the arc *Bm*.

Then $\sin A = \frac{a}{h}$, . . . (79) or $a = h \sin A$, ... (80)

and $\cos A = \frac{b}{h}$, . . . (81) or $b = h \cos A$. . . . (82)

With a radius *AH* describe the arc *Hn*; and we have

$\tan A = \frac{a}{b}$, . . . (83) or $a = b \tan A$, . . . (84)

and $\cot A = \frac{b}{a}$, . . . (85) or $b = a \cot A$. . . . (86)

From Geometry, we have $h^2 = a^2 + b^2$, . . . (87)

$A + B = 90°$, . . . (88) $\Delta = \frac{1}{2} ab$, ... (89)

From (80) and (82) we see that, *in any right plane triangle, either leg* is equal to the hypothenuse multiplied by the sine of the opposite angle, or by the cosine of the adjacent angle.*

From (84) and (86) we have: *Either leg is equal to the other leg multiplied by the tangent of the angle opposite to the first leg, or by the cotangent of the angle adjacent to it.*

---

* The two sides about the right angle are called legs.

Solution of Right Plane Triangles.

82. Equations (80), (82), (84), (86), (87) and (88) express relations between the sides and functions of the acute angles in any right plane triangle. They are simultaneous, but not all independent. (80), (82) and (88) are independent. Combining so as to eliminate A or B, there will remain two equations containing a, b, h, and a function of one acute angle. Hence, two of the quantities must be known, one of which must be a side, in order to determine the others.

Four cases arise; there may be given:

I. Either acute angle and the hypothenuse.
II. Either acute angle and either leg.
III. The hypothenuse and either leg.
IV. The two legs.

General Rule. Express each required part in terms of the two given parts only, by means of the formulas § 81; *then determine the required parts by use of tables.*

Some exceptions to the use of this rule arise when the formulas resulting from its application are not convenient for logarithmic computations.

83. Case I. *Given either acute angle and the hypothenuse.*

Examples.

Given $A = 27° \, 34'$, $h = 325$ ft., find the remaining parts.

From (88) we have $B = 90° - A = 62° \, 26'$.

To find a and b:

		Check.
$a = h \sin A$	$b = h \cos A$	$a = b \tan A$
$\log a = \log h + \log \sin A$	$\log b = \log h + \log \cos A$	$\log a = \log b + \log \tan$
$\log h = 2.5118\hat{8}$	$\log h = 2.5118\hat{8}$	$\log b = 2.45955$
$\log \sin A = 9.6653\hat{7} - 10$*	$\log \cos A = 9.9476\hat{6} - 10$	$\log \tan A = 9.71771 - 10$
$\log a = 2.17726$	$\log b = 2.45955$	$\log a = 2.17726$
$a = 150.404$ ft.	$b = 288.1$ ft.	$a = 150.404$ ft.

---

* The -10 corrects the tabular log sin, log cos, etc., which, in general, are 10 too large.

Hereafter the -10 will not be written, but each final log must be corrected, if necessary, before pointing off the corresponding number.

2. Given $h = 643.62$ ft., $A = 84° 40' 35''$, find the remaining parts. *Ans.* $B = 5° 19' 25''$, $a = 640.86$ ft., $b = 59.716$ ft.

3. Given $h = 125.7$ yds., $B = 75° 12'$, find the remaining parts. *Ans.* $A = 14° 48'$, $a = 32.11$ yds., $b = 121.5\dot{3}$ yds.

84. Case II. *Given either leg and either acute angle.*

Examples.

1. Given $a = 56.293$ yds., $A = 54° 27' 39''$, find B, b and h. *Ans.* $B = 35° 32' 21''$, $b = 40.211$ yds., $h = 69.178$ yds.

2. Given $b = 984.27$ metres, $B = 22° 59' 11''$, find $A = 67° 0' 49''$, $h = 2520.5$ metres, $a = 2320.3$ metres.

85. Case III. *Given the hypothenuse and either leg.*

Let b and h be given; then, (81) and (88),

$$\cos A = \frac{b}{h} = \sin B.$$

Knowing either A or B, the other is its complement.

From (87), $$a = \sqrt{(h + b)(h - b)}.$$

Examples.

1. Given $h = 2391.8$ yds., $a = 385.7$ yds., find $A = 9° 16' 48''$, $B = 80° 43' 12''$, $b = 2360.5$ yds.

2. Given $h = 9$ metres, $b = 8$ metres, find $A = 27° 16''$, $B = 62° 44'$, $a = 4.123$ metres.

When the given leg b is nearly equal to h, A is near 0° and B near 90°; and neither will be accurately determined by the above method. (79) or (83) may then be used; or,

From (45) and (48), we have

$$\sin\frac{A}{2} = \sqrt{\frac{1 - \cos A}{2}}, \qquad \tan\frac{A}{2} = \sqrt{\frac{1 - \cos A}{1 + \cos A}}.$$

Substituting $\frac{b}{h}$ for cos A, and reducing, we have

$$\sin\frac{A}{2} = \sqrt{\frac{h - b}{2h}}, \quad . \quad (90) \qquad \tan\frac{A}{2} = \sqrt{\frac{h - b}{h + b}}. \quad . \quad (91)$$

From either of which $\frac{A}{2}$ may be more readily computed, when a is not required.

Examples.

1. Given $b = 20.15$ yds., $h = 20.75$ yds.,

find $A = 13° \; 48' \; 43''$.

2. Given $b = 737.24$ metres, $h = 738.04$ metres.

Ans. $A = 2° \; 40' \; 05''.0$.

86. **Case IV.** *Given the two legs.* From (84) and (86),

$$\tan A = \frac{a}{b} = \cot B;$$

and from (87), $h = \sqrt{a^2 + b^2}$.

From (79) and (81,)

$$h = \frac{a}{\sin A} \quad \text{and} \quad h = \frac{b}{\cos A},$$

which are more convenient for applying logarithms, and the two values will check all results.

Examples.

1. Given $a = 43$ yds., $b = 55$ yds.

Ans. $A = 38° \; 1' \; 8''$, $B = 51° \; 58' \; 52''$, $h = 69.81$ yds.

2. Given $a = 751.8$ metres, $b = 534.2$ metres.

Ans. $A = 54° \; 36' \; 14''$, $B = 35° \; 23' \; 46$, $h = 922.3$ metres.

Miscellaneous Examples.

Given.		*Required.*	
$h = 361.4$ ft.,	$A = 41^\circ\ 12'$,	$a = 238.05$ ft.,	$b = 271.92$ ft., $B = 48^\circ\ 48'$.
$h = 27.18$ ft.,	$A = 26^\circ\ 18'\ 05''$,	$a = 12.043$ ft.,	$b = 24.366$ ft., $B = 63^\circ\ 41'\ 55''$.
$h = 749$ yds ,	$A = 47^\circ\ 03'\ 10''$,	$B = 42^\circ\ 56'\ 50''$,	$a = 548.25$ yds., $b = 510.31$ yds.
$h = 439$ metres,	$B = 62^\circ\ 21'\ 10''$,	$A = 27^\circ\ 38'\ 50''$,	$a = 203.708$ metres $b = 388.87$ metres
$b = 2880$ mi.,	$A = 53^\circ\ 8'$,	$B = 36^\circ\ 52'$,	$a = 3840.45$ mi., $h = 4800.33$ mi.
$a = 125.694$ ft.,	$A = 81^\circ\ 15'\ 07''$,	$B = 8^\circ\ 44'\ 53''$,	$b = 19.342$ ft., $h = 127.174$ ft.
$a = 150.404$ metr.,	$B = 62^\circ\ 26'$,	$A = 27^\circ\ 34'$,	$b = 288.1$ metres, $h = 325$ metres.
$a = 7.5$ metres,	$h = 12.5$ metres,	$b = 10$ metres,	$B = 53^\circ\ 07'\ 48''$, $A = 36^\circ\ 52'\ 12''$.
$b = 473.8$ yds.,	$h = 643.7$ yds.,	$a = 435.73$ yds.,	$A = 42^\circ\ 36'\ 13''$, $B = 47^\circ\ 23'\ 47''$.
$b = 1277$ ft.,	$h = 2132$ ft.,	$a = 1707.24$ ft.,	$A = 53^\circ\ 12'\ 14''$, $B = 36^\circ\ 47'\ 46''$.
$a = 26$ yds.,	$b = 15$ yds.,	$A = 60^\circ\ 01'\ 06''$,	$B = 29^\circ\ 58'\ 54''$, $h = 30.016$ yds.
$a = 347.21$ ft.,	$b = 1052.29$ ft.,	$A = 18^\circ\ 15'\ 38''$,	$B = 71^\circ\ 44'\ 22''$, $h = 1108.09$ ft.
$a = 118.297$ yds.,	$b = 122.416$ yds.	$A = 44^\circ\ 01'\ 10''$,	$B = 45^\circ\ 58'\ 50''$, $h = 170.234$ yds.

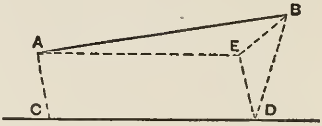
PROJECTIONS OF STRAIGHT LINES AND PLANE FIGURES.

87. The projection of a point on a line is the foot of a perpendicular through the point to the line.

The projection of a point on a plane is the point where a perpendicular, through the point to the plane, pierces the plane.

The projection of a straight line of limited length on a second straight line is that part of the second line which lies between the projections of the extreme points of the first line upon the second; and *it is equal to the length of the first line multiplied by the cosine of the angle which the two lines make with each other.*

To prove this, let AB be any straight line of limited length, and CD any straight line on which it is to be projected. Through A draw AC perpendicular, and AE parallel to CD. The angle BAE will be that which the two lines AB and CD make with each other. Through B pass a plane perpendicular to CD, cutting AE in E, and CD in D. Draw BE, ED and DB.



Then CD is the required projection of AB.

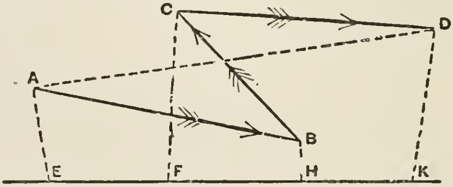
AC and ED are parallel, because they lie in the plane of the two parallels AE and CD, and both are perpendicular to CD. Hence, $ACDE$ is a parallelogram, and $AE = CD$. But

$$AE = AB \cos BAE, \text{ since } BEA = 90^\circ.$$

Therefore,
$$CD = AB \cos BAE.$$

88. *The algebraic sum of the projections on a given straight line, of a series of straight lines connecting any two points, is equal to the projection, on the same line, of the straight line joining the two points.*

Let AB, BC and CD be a series of straight lines connecting the points A and D, and AD a straight line joining the two points. Let EK be the straight line on which the projections are to be made. Through A, B, C and D, respectively, draw AE, BH, CF and DK perpendicular to EK.



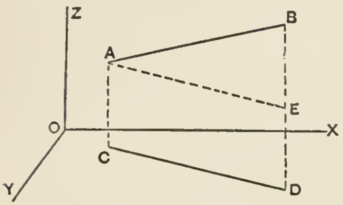
Then, the algebraic sum of the projections of AB, BC and CD upon EK is $EH - HF + FK$. HF is negative because the angle made by BC with EK, as estimated, is obtuse, and its cosine, therefore, negative.

EK is the projection of AD on EK, and

$$EK = EH - HF + FK.$$

In a similar manner, the principle may be proved for any other series of lines.

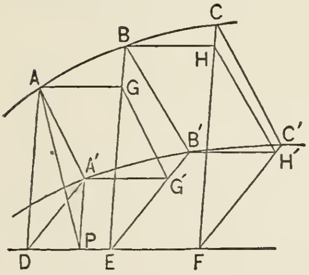
89. *The projection of a straight line of limited length on any plane, is equal to the length of the line multiplied by the cosine of the angle which the line makes with the plane.*



Let AB be any limited straight line, and XOY any plane. Through A and B, respectively, draw the straight lines AC and BD perpendicular to the plane XOY, piercing it in the points C and D. Join C and D by the straight line CD; it is the required projection. Through A draw AE parallel to CD. The angle BAE is that which the line AB makes with the plane XOY; and since $ACDE$ is a parallelogram, we have

$$CD = AE = AB \cos BAE.$$

90. Let $DACBF$ be a plane figure, and $DA'B'C'F$ its projection on another plane intersecting the first in the right line DF.



Assume any number of points on DF, through which draw right lines parallel to DA and DA' respectively. Through the points in which the first set intersect the curve ABC, and the points in which the second set intersect $A'B'C'$, draw right lines parallel to DF, forming the two sets of parallelograms AE, BF, etc., and $A'E$, $B'F$, etc.

Through AA', the projecting line of A, pass a plane perpendicular to DE cutting the two planes in the right lines AP and PA' respectively.

The angle APA', which we will denote by θ, is that made by the planes with each other

$$A'P = AP \cos \theta; \qquad \text{area } AE = AP \times DE;$$

$$\text{area } A'E = A'P \times DE = AP \cos \theta \times DE = \text{area } AE \cos \theta.$$

Similarly, each parallelogram in projection is equal to the corresponding one in the given figure, multiplied by cos θ.

Let n denote the number of parallelograms; then as the number of points on DF is increased, we have

$$DA'B'C'F = \underset{n=\infty}{\text{limit}} \, (A'E + B'F + \text{etc.}) = \underset{n=\infty}{\text{limit}} \, (AE + BF + \text{etc.}) \cos\theta$$
$$= DABCF \cos\theta.$$

In a similar manner, it may be shown that *the area of the projection of any plane figure is equal to that of the given figure multiplied by the cosine of the angle made by its plane with the plane of projection.*

Definitions.

91. Considering any place on the earth's surface and the corresponding radius of the earth:

A Horizontal Plane is one which is perpendicular to the radius.

A Vertical Plane is one which contains the radius.

A Horizontal Line is one which is perpendicular to the radius.

A Vertical Line is one which is parallel to the radius.

An Oblique Line is one which is neither perpendicular nor parallel to the radius.

A Horizontal Angle is one whose sides are horizontal.

A Vertical Angle is one, the plane of whose sides is vertical.

An Angle of Elevation is a vertical angle having one of its sides horizontal and the other above the horizontal side.

An Angle of Depression is a vertical angle having one side horizontal and the other below the horizontal side.

An Oblique Angle is one, the plane of whose sides is neither horizontal nor vertical.

Applications.

1. A railway 463 ft. 3 in. in length has a uniform grade of 3°. Find the total rise. *Ans.* 24 ft. 3 in.

2. A railway with a uniform grade rises 150 ft. in 3000 ft. Find the grade. *Ans.* $\sin^{-1}(\frac{1}{20}) = 2° \; 51' \; 58''$.

3. A ladder 38 ft. in length, placed in a street, touches a point 29 ft. 6 in. in height on a vertical wall on one side of the street, or a point 28 ft. in height on the opposite wall. Find the width of the street. *Ans.* 49.643 ft.

4. The angle of elevation to the top of a steeple, measured from a point 200 ft. from its base in a horizontal direction, is 47° 30′. What is the height of the steeple? *Ans.* 218.26 ft.

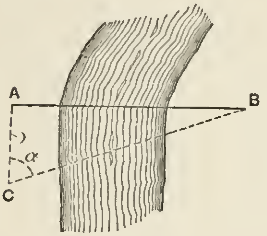
5. A tower 103 ft. in height casts a shadow upon the horizontal plane of its base, 51.5 ft. in length. Find the angle of elevation of the sun.

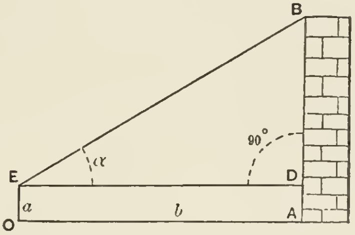
Ans. $63° 23' 06''$.

6. A and B are points on opposite sides of a river. Find the distance AB from the A side.

Method. Lay off $AC \perp AB$. Measure AC and the angle $ACB = \alpha$.

Given $AC = 80$ yds., $\alpha = 85° 22' 30''$, find $AB = 989.1$ yds.





7. Find the height of the vertical wall AB when $AO = DE = b = 400$ ft. $DA = EO = a = 6$ ft., and $\alpha = 22° 17'$. *Ans.* 169.9 ft.

8. Assuming the earth to be a sphere with a radius of 3963 mi., find the height of a light-house light just visible from a point 20 mi. at sea. *Ans.* 266.5 ft.

Suggestion. The line of sight is tangent, at the point of sight, to the great circle through it and the foot of the light-house. It forms one side of a right triangle whose hypothenuse exceeds the radius of the earth by the required height. The angle at the centre of the earth should be found from its radian measure. Compute log h by (81), then from (90) find $h - b$.

9. Assuming the height of the great pyramid to be 486 ft., and the radius of the earth to be 3960 mi., from what distance can the top of the pyramid be seen?

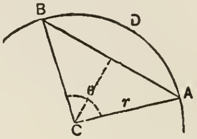
Ans. 27 mi.

10. Assuming the radius of the earth to be 3963 mi., the longitudes of Columbia, S. C., and Los Angelos, Cal., to be 81° 10′ W. and 118° 30′ W., respectively, and their common latitude to be 34° N.; find in miles the length of the arc of latitude which joins them. *Ans.* 2140.8 mi.

Suggestion. Find the length of the radius of the circle of latitude; then, with the difference in longitude, determine the required distance.

11. From a point 108 ft. above a horizontal plane the angles of depression of the top and bottom of a tower standing on the plane are 30° and 60°, respectively. Find the height of the tower. *Ans.* 72 ft.

12. **Area of a Circular Segment.** Let ABD be any circular segment in a circle whose radius is r. Denote the corresponding angle at the centre by θ.



From the figure, we have

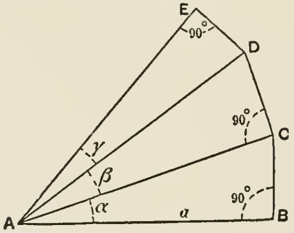
$$\text{seg. } ABD = \text{sector } CADB - \text{triangle } CAB.$$

From Geometry, and § 64, we have

$$\text{sec. } CADB = \frac{r^2\theta}{2}, \quad \text{and} \quad \text{triangle } CAB = r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{r^2 \sin\theta}{2}.$$

Hence, $$\text{segment} = \frac{r^2\theta}{2} - \frac{r^2\sin\theta}{2} = \frac{r^2}{2}(\theta - \sin\theta).$$

13. Express AE and ED, in terms of $a = AB$, and functions of α, β and γ.
Method. Find AC, then AD.



14. Same, in terms of $b = BC$, and functions of α, β and γ.

CHAPTER VIII.

OBLIQUE PLANE TRIANGLES.

92. Any oblique plane triangle can be solved by using two auxiliary right plane triangles formed by drawing a perpendicular from a vertex to the opposite side. Such a vertex should, if possible, be selected so that one of the auxiliary triangles will have one side and one other part given in addition to the right angle. The right triangles may then be solved, and the required parts of the oblique triangle found. If the three sides are given, no vertex will fulfil the above condition. In this case, P. XII. B. IV. Davies' Geo. gives data for the solution.

In general, it is more convenient to solve an oblique triangle directly from the relations between its parts.

RELATIONS BETWEEN THE PARTS OF OBLIQUE PLANE TRIANGLES

93. *Relation of Angles.* Let ABC be any plane triangle, in which the angles are denoted by A, B and C, respectively.

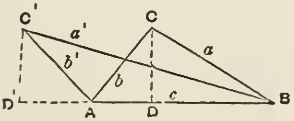
By Geometry,

$$A + B + C = 180^\circ. \quad . \; . \; . \; . \; . \; . \; . \quad (92)$$

94. *Area in Terms of Two Sides and their Included Angle.* Denote by a, b, c, the sides opposite to A, B, C, respectively, and by Δ, the area. From the vertex C draw CD perpendicular to AB. Then

$$\Delta = \tfrac{1}{2} AB \times DC,$$

and $\sin A = \dfrac{DC}{AC}$, or $DC = b \sin A$,



which, when substituted above, gives

$$\Delta = \tfrac{1}{2}bc \sin A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (93)$$

That is: *The area of any plane triangle is equal to half the rectangle of any two sides multiplied by the sine of their included angle.*

95. *Relations of Opposite Parts.* Three different expressions for the area of *ABC* may be written, giving

$$\tfrac{1}{2}bc \sin A = \tfrac{1}{2}ac \sin B = \tfrac{1}{2}ab \sin C.$$

Divide each member by $\frac{1}{2}abc$, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (94)$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (95)$$

That is: *The sides of any plane triangle are proportional to the sines of their opposite angles.*

Equation (95) may be written in the form

$$a : b :: \sin A : \sin B;$$

whence, by composition and division.

$$a + b : a - b :: \sin A + \sin B : \sin A - \sin B;$$

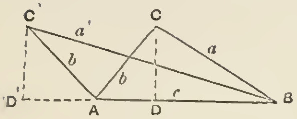
but from (66),

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)};$$

$$\therefore \; a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B). \quad . \qquad (96)$$

That is: *The sum of any two sides of a plane triangle is to their difference, as the tangent of one half the sum of their opposite angles is to the tangent of one half their difference.*

96. *Square of Any Side in Terms of the Other Sides and their Included Angle.*



If A is acute, D lies on the side of A toward B, or on the positive side, and we have

$$a^2 = b^2 + c^2 - 2cAD.$$

If A is obtuse, we have

$$a'^2 = b'^2 + c^2 + 2cD'A,$$

D' being on the negative side of A. Neglecting the dashes, both may be written,

$$a^2 = b^2 + c^2 - 2cAD.$$

Taking $AC = b = R$, we have

$$\cos A = \frac{AD}{R}, \qquad \text{or} \qquad AD = b \cos A.$$

Hence,

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (97)$$

97. *Functions of Angles in Terms of the Sides.* From (97) we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

which is inconvenient for logarithmic computation. By adding unity to both members, we have

$$1 + \cos A = \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c + a)(b + c - a)}{2bc};$$

and subtracting each member of (98) from unity,

$$1 - \cos A = \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc}$$

Hence, by (45), (47) and (48),

$$\sin \tfrac{1}{2}A = \sqrt{\frac{1 - \cos A}{2}} = \sqrt{\frac{(a + b - c)(a - b + c)}{4bc}},$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{1 + \cos A}{2}} = \sqrt{\frac{(b + c + a)(b + c - a)}{4bc}},$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \sqrt{\frac{(a + b - c)(a - b + c)}{(b + c + a)(b + c - a)}},$$

which are convenient for logarithmic computation To make these forms more compact, let

$$a + b + c = s, \qquad \text{then} \qquad a + b - c = s - 2c;$$

$$a - b + c = s - 2b, \qquad b + c - a = s - 2a;$$

whence

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - c)}{bc}}, \quad . \; . \; . \; . \; . \quad (99)$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{\tfrac{1}{2}s(\tfrac{1}{2}s - a)}{bc}}, \quad . \; . \; . \; . \; . \; . \; . \quad (100)$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - c)}{\tfrac{1}{2}s(\tfrac{1}{2}s - a)}}. \quad . \; . \; . \; . \; . \quad (101)$$

These values are all positive, since $A < 180°$, and $\tfrac{1}{2}A < 90°$. (101) will usually be preferred for determining A.

If two or more angles are required in the same triangle, the following will be convenient. Let

$$K = \sqrt{\frac{(\tfrac{1}{2}s - a)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - c)}{\tfrac{1}{2}s}} \quad . \; . \; . \quad (102)$$

This reduces (101) to the form

$$\tan \tfrac{1}{2}A = \frac{K}{\tfrac{1}{2}s - a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

By analogy, since K is symmetrical with respect to a, b and c,

$$\tan \tfrac{1}{2}B = \frac{K}{\tfrac{1}{2}s - b},$$

$$\tan \tfrac{1}{2}C = \frac{K}{\tfrac{1}{2}s - c}.$$

98. *Area in Terms of the Sides.* We have, by (93),

$$\varDelta = \tfrac{1}{2}bc \sin A.$$

By (36), making $2\alpha = A$, we have

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A,$$

which, by (99) and (100), gives

$$\sin A = \frac{2\sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s - a)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - c)}}{bc}. \quad . \quad . \quad . \quad (104)$$

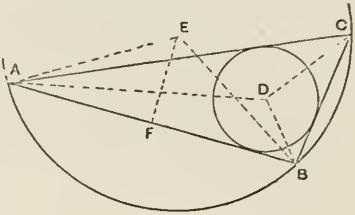
Hence, (93), $\varDelta = \sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s - a)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - c)}$. . . . (105)

Comparing (105) with (102), we have

$$\varDelta = \tfrac{1}{2}sK, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

which is more convenient than (105), if K has already been computed.

99. *Expressions for Radii of Circumscribed and Inscribed Circles.*



Denote the radii by R and r, respectively. Let E and D, be their respective centres. Draw the right lines AE, BE, AD, BD, CD, and from E draw EF perpendicular to AB.

To find R. The angle ACB being inscribed, we have

$$C = ACB = \tfrac{1}{2}AEB = FEB,$$

$$\sin FEB = \frac{FB}{R} = \frac{\frac{1}{2}c}{R} = \sin C.$$

Hence,
$$R = \frac{\frac{1}{2}c}{\sin C}. \quad \dots \dots \dots \quad (107)$$

Compare (95).

From § 94, we have $\sin C = \dfrac{\Delta}{\frac{1}{2}\,ab}$.

Hence, (107),
$$R = \frac{abc}{4\Delta}. \quad \dots \dots \dots \quad (108)$$

To find r. The triangle ABC is composed of ABD, BCD, CAD, which have the common altitude r.

Hence,
$$\Delta = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = \tfrac{1}{2}(a + b + c)\,r = \tfrac{1}{4}sr,$$

and
$$r = \frac{\Delta}{\frac{1}{2}s}; \quad \dots \dots \dots \quad (109)$$

or by (106),
$$r = K = \sqrt{\frac{(\frac{1}{2}s - a)(\frac{1}{2}s - b)(\frac{1}{2}s - c)}{\frac{1}{2}s}}. \quad \dots \quad (110)$$

100. Equations (92), (95), (96), (97), (98), (99), (100), (101), (102), express relations between the parts A, B, C, a, b, c.

To show that this group contains but three independent equations.

From the methods of deduction, (96), (98), (99), (100), (101) and (102) depend on (92), (95) and (97).

From (92),
$$A + B + C = 180°.$$

From (95),
$$\begin{cases} \dfrac{a}{\sin A} = \dfrac{b}{\sin B}, \\[2ex] \dfrac{a}{\sin A} = \dfrac{c}{\sin C}. \end{cases}$$

From (97)
$$a^2 = b^2 + c^2 - 2bc \cos A.$$

It remains to be shown that (97) is dependent on the other three.

Evidently,
$$a^2 = \frac{a^2}{\sin^2 A} \sin^2 A.$$

But
$$A = 180° - (B + C),$$

and by (28),

$$\sin A = \sin(B + C) = \sin B \cos C + \cos B \sin C;$$
$$\begin{aligned} \sin^2 A &= \sin^2 B(1 - \sin^2 C) + 2 \sin B \cos C \cos B \sin C + (1 - \sin^2 B) \sin^2 C \\ &= \sin^2 B + \sin^2 C + 2 \sin B \sin C[\cos B \cos C - \sin B \sin C]. \end{aligned}$$

But by (30), $\cos(B + C) = \cos B \cos C - \sin B \sin C$;
and by (92), $\cos(B + C) = \cos[180° - A] = -\cos A$.

Hence,
$$\sin^2 A = \sin^2 B + \sin^2 C - 2 \sin B \sin C \cos A,$$

and
$$a^2 = \frac{a^2 \sin^2 A}{\sin^2 A} = \frac{a^2 \sin^2 B}{\sin^2 A} + \frac{a^2 \sin^2 C}{\sin^2 A} - 2 \frac{a \sin B}{\sin A} \cdot \frac{a \sin C}{\sin A} \cos A;$$

which, by (95), gives (97), which therefore depends on the other three.

In the above three equations, from (92) and (95), we may, by (92), eliminate any angle as C, giving a group of two simultaneous and independent equations containing five elements, A, B, a, b, c, any three of which will determine the other two.

Hence, as in Geometry, any three elements which include at least one side will completely determine the triangle.

SOLUTION OF OBLIQUE PLANE TRIANGLES.

101. There are four cases. There may be given:

I. One side and two angles.
II. Two sides and an angle opposite to one of them.
III. Two sides and their included angle.
IV. Three sides.

In general, express each required part separately in terms of the given parts only. From these expressions make the computations.

102. CASE I. *Given one side and two angles.* Let a, A and B be given.

To find C. From (92), we have

$$C = 180^\circ - (A + B).$$

To find b and c. From (95), we have

$$b = \frac{a \sin B}{\sin A}, \quad c = \frac{a \sin C}{\sin A} \quad \text{and} \quad c = \frac{b \sin C}{\sin B}.$$

The double computation of c will serve as a check on those of b and c.

If the area is required, eliminate b from (93) by (95), giving

$$\varDelta = \frac{1}{2} \cdot \frac{a^2 \sin B \sin C}{\sin A}. \quad \ldots\ldots \quad (111)$$

Examples.

1. Given $a = 250.4$ yds., $A = 72° 44' 05''$, $B = 15° 19' 50''$.
To find C. $C = 180° - (A + B) = 91° 56' 05''$.
To find b and c.

$$b = \frac{a \sin B}{\sin A}, \qquad c = \frac{a \sin C}{\sin A}.$$

$$\log b = \log a - \log \sin A + \log \sin B,$$
$$\log c = \log a - \log \sin A + \log \sin C.$$

$\log a$	$2.3986\hat{3}$
a. c. $\log \sin A$	0.02002
$+ \log \sin B$	$9.4222\hat{4}$
$\log b$	1.84090
$b = 69.327$ yds.	

$\log a$	$2.3986\hat{3}$
a. c. $\log \sin A$	0.02002
$+ \log \sin C$	$9.9997\hat{5}$
$\log c$	2.41841
$c = 262.07$ yds.	

To check the work, and find the area:

$$c = \frac{b \sin C}{\sin B}, \quad \Delta = \frac{1}{2} \frac{a^2 \sin B \sin C}{\sin A}.$$

$$\log c = \log b - \log \sin B + \log \sin C,$$
$$\log \Delta = 2 \log a - \log \sin A - \log 2 + \log \sin B + \log \sin C.$$

$\log b$	1.84090
a. c. $\log \sin B$	$0.5777\hat{5}$
$+ \log \sin C$	$9.9997\hat{5}$
$\log c$	2.41841

$2 \log a$	4.79727
a. c. $\log \sin A$	0.02002
a. c. $\log 2$	9.69897
$+ \log \sin B$	$9.4222\hat{4}$
$+ \log \sin C$	$9.9997\hat{5}$
$\log \Delta$	3.93826
$\Delta = 8674.8$ sq. yds.	

2. Given $A = 57° 35'$, $B = 64° 51'$, $c = 600$ ft.
Ans. $C = 57° 34'$, $a = 600.11$ ft., $b = 643.49$ ft.

3. Given $B = 83° 52' 12''$, $C = 57° 42' 43''$, $b = 400$ metres.
Ans. $A = 38° 25' 05''$, $a = 249.98$ m., $c = 340.09$ m.

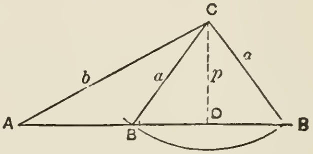
4. Given $B = 54° 18' 07''$, $C = 93° 23' 43''$, $c = 60.231$ yds.
Ans. $A = 32° 18' 10''$, $a = 32.244$ yds., $b = 49.$ yds.

5. Given $B = 125° 41' 53''$, $C = 21° 59' 57''$, $c = 22.602$ yds.
Ans. $A = 32° 18' 10''$, $a = 32.244$ yds., $b = 49.$ yds.

103. CASE II. *Given two sides and an angle opposite one of them.*

Let a, b and A be given.

Construct a figure as indicated, and denote by p the perpendicular DC.



1st. If $A > 90°$, or if $A = 90°$, but one solution is possible, and there can be no solution unless $a > b$.

2d. If $A < 90°$, four cases arise:

When $a > b$ or $a = b$, there is one solution;
When $b > a > p$, there are two solutions;
When $a = p$, there is one solution;
When $a < p$, there is no solution.

Before beginning the computations, notice whether or not

$$A > 90° \quad \text{or} \quad A = 90°, \quad \text{and} \quad a < b \quad \text{or} \quad a = b.$$

If so, no solution is possible; if not, from (95) write:

$$\sin B = \frac{b \sin A}{a},$$

$$\log \sin B = \log b + \log \sin A - \log a.$$

Compute $\log \sin B$. There are four cases:

1st. If $\log \sin B > 0$, $\sin B > 1$, which is impossible; therefore no solution.

2d. If $\log \sin B = 0$, $\sin B = 1$, $B = 90^\circ$, and $a = p$, there will be one solution only, provided $A < 90^\circ$; otherwise none.

3d. If $\log \sin B < 0$, $\sin B < 1$ and two values of $B < 180^\circ$, will result, viz., $B < 90^\circ$, and $B' = 180^\circ - B$.

When $A > 90^\circ$ or $A = 90^\circ$, but one value, $B < 90^\circ$, is admissible.

When $A < 90^\circ$, both, $B < 90^\circ$, and $B' > 90^\circ$, are required.

Having found B and B', the other required elements can be found as in § 81, C and C' from (92), c and c' from (95).

Examples.

1. Given $a = 32.244$ yds., $b = 49$ yds., $A = 32^\circ\ 18'\ 10''$.

To find B. $$\sin B = \frac{b \sin A}{a},$$

$$\log \sin B = \log b + \log \sin A - \log a.$$

$$\begin{aligned} \log b &= 1.6901\hat{9} \\ \log \sin A &= 9.72786 \\ \text{a.c.} \log a &= 8.49155 \\ \hline \log \sin B &= 9.9096\hat{6} \\ &\ 54^\circ\ 18'\ 03'' \end{aligned}$$

$$\begin{aligned} \therefore B &= 54^\circ\ 18'\ 03'' \\ B' &= 125^\circ\ 41'\ 57'' \end{aligned}$$

To find C. $$\begin{aligned} C &= 180^\circ - (A + B) = 93^\circ\ 23'\ 47'', \\ C' &= 180^\circ - (A + B') = 21^\circ\ 59'\ 53''. \end{aligned}$$

To find c. $$c = \frac{a \sin C}{\sin A} = \frac{b \sin C}{\sin B},$$

$$\begin{aligned} \log c &= \log a - \log \sin A + \log \sin C \\ &= \log b - \log \sin B + \log \sin C. \end{aligned}$$

	Check.
$\log a = 1.50845$	$\log b = 1.6901\hat{9}$
a. c. $\log \sin A = 0.27214$	a. c. $\log \sin B = 0.0903\hat{9}$
$\log \sin C = 9.9992\hat{3}$	$9.9992\hat{3}$
$\log \sin C' = 9.57354$	9.57354
$\log c = 1.7798\hat{2}$	$1.7798\hat{2}$
$\log c' = 1.35413$	1.35413

$c = 60.231$ yd., $c' = 22.601$ yd.

2. Given $a = 32.244$ ft., $b = 49$ ft., $A = 147° 41' 50''$.

Ans. Impossible by inspection.

3. Given $a = 45.90$ ft., $b = 34.217$ ft., $B = 39° 8' 20''$.

$$\textit{Ans.} \begin{cases} A = 57° 51' 22'',\ C = 83° 0' 18'',\ c = 53.806 \text{ ft.} \\ A' = 122° 8' 38'',\ C' = 18° 43' 2'',\ c' = 17.395 \text{ ft.} \end{cases}$$

4. Given $a = 22.14$ ft., $b = 49.50$ ft., $B = 97° 23' 10''$.

Ans. $A = 26° 19' 53''$, $C = 56° 16' 57''$, $c = 41.516$ ft.

5. Given $b = 40.39$ yds., $c = 50.48$ yds., $B = 63° 15' 10''$.

Ans. Impossible.

6. Given $a = 248$ miles, $c = 354$ miles, $A = 41° 36'$.

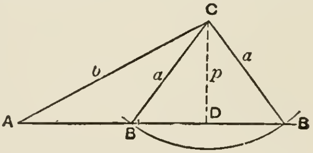
$$\textit{Ans.} \begin{cases} C = 71° 23' 22'',\ B = 67° 0' 38'',\ b = 343.87 \text{ mi.} \\ C' = 108° 36' 38'',\ B' = 29° 47' 22'',\ b' = 185.577 \text{ mi.} \end{cases}$$

7. Given $a = 216.09$ metres, $c = 191.55$ metres, $C = 62° 25' 52''$.

Ans. $A = 90°$, $B = 27° 34' 08''$, $b = 100.007$ metres.

8. Given $a = 150.404$ ft., $b = 288.1$ ft., $A = 27° 34'$.

$$\textit{Ans.} \begin{cases} B = 62° 26',\ C = 90°,\ c = 325 \text{ ft.} \\ B' = 117° 34',\ C' = 34° 52',\ c' = 185.79 \text{ ft.} \end{cases}$$



104. *Special Method.* The perpendicular $DC = p$ determines two right triangles, ADC and BDC. In ADC, b and A are given, and p may be computed. Then, in BDC, a and p are known, and the angle DCB may be computed. $B = 90° \mp DCB$ according as B is acute or obtuse. c may then be computed as before (§ 103). This method does not, in general, increase the accuracy.

Examples.

1. Given $a = 57$ yds., $b = 69$ yds., $A = 54° 27' 39''$, find B accurately. *Ans.* $B = 80° 4' 30''$, $B' = 99° 55' 30''$.

2. Given $a = 56$ ft., $b = 55$ ft., $A = 90° 20' 30''$, find B accurately. *Ans.* $B = 79° 9' 2''$.

105. CASE III *Given two sides and their included angle.*
Let a, b and C be given.
To find A and B.

From (92), $$A + B = 180° - C.$$

By (§ 95), $a + b : a - b :: \tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$,

$$\tan \tfrac{1}{2}(A - B) = \frac{(a - b) \tan \frac{1}{2}(A + B)}{a + b};$$

from which $\frac{1}{2}(A - B)$ should be computed. $\frac{1}{2}(A - B) < 90°$, since $\frac{1}{2}A < 90°$, and $\frac{1}{2}B < 90°$. If $a > b$, we have $A > B$; if $a = b$, $A = B$; if $a < b$, $A < B$.

$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B), \quad \text{and} \quad B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B).$$

To find c. The angles being known, c may be computed from (95), as in Case I. Double computation checks the preceding work.

Examples.

1. Given $a = 18\ 739$ metres, $b = 7.642$ metrse. $C = 42° \ 18' \ 28''$. *To find A and B.*

$$\tan \tfrac{1}{2}(A - B) = \frac{(a - b) \tan \frac{1}{2}(A + B)}{a + b}.$$

$a - b$	$= 11.097$	$\log (a - b)$	$= 1.0452\hat{0}$
$a + b$	$= 26.381$	a. c. $\log (a + b)$	$= 8.57871$
$\frac{1}{2}(A + B)$	$= 90° - \frac{1}{2}C$	$+ \log \tan \frac{1}{2}(A + B)$	$= 0.41235$
	$= 68° \ 50' \ 46''$		
		$\log \tan \frac{1}{2}(A - B)$	$0.0362\hat{6}$

$\frac{1}{2}(A - B) = 47° \ 23' \ 22''$

$$\therefore A = 116° \ 14' \ 08'', \qquad B = 21° \ 27' \ 24''$$

To find c. $$c = \frac{a \sin C}{\sin A} = \frac{b \sin C}{\sin B}.$$

$\log a$	$1.2727\hat{4}$	$\log b$	$0.8832\hat{0}$
a. c. $\log \sin A$	$0.0472\hat{1}$	a. c. $\log \sin B$	0.43676
$\log \sin C$	9.82809	$\log \sin C$	9.82809
$\log c$	1.14805	$\log c$	$.1480\hat{5}$

$\therefore c = 14.062.$

2. Given $a = 400$ yds., $c = 534.54$ yds., $B = 6° \; 27' \; 18''$.
Ans. $A = 18° \; 9' \; 45''$, $C = 155° \; 22' \; 57''$, $b = 144.265$ yds.

3. Given $b = 400$ miles, $c = 368.2$ miles, $A = 12° \; 49' \; 04''$.
Ans. $B = 103° \; 49' \; 19''$, $C = 63° \; 21' \; 37''$, $a = 91.39$ miles.

106. CASE IV. *Given the three sides a, b, c.*
To find A, B and C.
From (§ 97) we have

$$\tan \tfrac{1}{2}A = \frac{K}{\frac{1}{2}s - a}, \qquad K = \sqrt{\frac{(\frac{1}{2}s - a)(\frac{1}{2}s - b)(\frac{1}{2}s - c)}{\frac{1}{2}s}}$$

$$\tan \tfrac{1}{2}B = \frac{K}{\frac{1}{2}s - b}, \qquad s = a + b + c,$$

$$\tan \tfrac{1}{2}C = \frac{K}{\frac{1}{2}s - c}.$$

Compute each angle separately, and check the results by (92), $A + B + C = 180°$.

Examples.

1. Given $a = 10$ ft., $b = 12$ ft., $c = 14$ ft.

$$s = a + b + c = 36 \text{ ft.} \qquad K = \sqrt{\frac{(\tfrac{1}{2}s - a)(\tfrac{1}{2}s - b)(\tfrac{1}{2}s - c)}{\tfrac{1}{2}s}}.$$

$$\tfrac{1}{2}s = 18 \text{ ft.}$$

$$\begin{array}{lll}
(\tfrac{1}{2}s - a) = 8 \text{ ft.} & \log(\tfrac{1}{2}s - a) & 0.90309 \\
(\tfrac{1}{2}s - b) = 6 \text{ ft.} & +\log(\tfrac{1}{2}s - b) & 0.77815 \\
(\tfrac{1}{2}s - c) = 4 \text{ ft.} & +\log(\tfrac{1}{2}s - c) & 0.60206 \\
 & \text{a. c. } \log \tfrac{1}{2}s & 8.7447\hat{2} \\
\hline
 & & 2)\underline{1.0280\hat{2}} \\
 & \log K & 0.5140\hat{1}
\end{array}$$

$$\tan \tfrac{1}{2}A = \frac{K}{\tfrac{1}{2}s - a}, \quad \tan \tfrac{1}{2}B = \frac{K}{\tfrac{1}{2}s - b}, \quad \tan \tfrac{1}{2}C = \frac{K}{\tfrac{1}{2}s - c}.$$

$$\begin{array}{rrr}
\log K = 0.5140\hat{1} & \log K = 0.5140\hat{1} & \log K = 0.5140\hat{1} \\
\text{a. c. } \log(\tfrac{1}{2}s - a) = 9.09691 & \text{a. c. } \log(\tfrac{1}{2}s - b) = 9.22185 & \text{a. c. } \log(\tfrac{1}{2}s - c) = 9.39794 \\
\hline
\log \tan \tfrac{1}{2}A = 9.6109\hat{2} & \log \tan \tfrac{1}{2}B = 9.7358\hat{6} & \log \tan \tfrac{1}{2}C = 9.9119\hat{5}
\end{array}$$

$$\tfrac{1}{2}A = 22^\circ\ 12'\ 27'', \quad \tfrac{1}{2}B = 28^\circ\ 33'\ 39'', \quad \tfrac{1}{2}C = 39^\circ\ 13'\ 54'',$$
$$A = 44^\circ\ 24'\ 54'', \quad B = 57^\circ\ 7'\ 18'', \quad C = 78^\circ\ 27'\ 48''.$$

Their sum is 180°. A small discrepancy may be due to inevitable tabular errors in the logarithmic values.

2. Given $a = 87$ yds., $b = 9$ yds., $c = 79$ yds.
Ans. $A = 151^\circ\ 16'\ 28''$, $B = 2^\circ\ 51'$, $C = 25^\circ\ 52'\ 32''$.

3. Given $a = 425.6$, $b = 679$, $c = 1034$.
Ans. $A = 16^\circ\ 6'\ 20''$, $B = 26^\circ\ 16'\ 6''$, $C = 137^\circ\ 37'\ 34''$.

Additional Examples.

1. Given $c = 400$ yds., $A = 38^\circ\ 25'\ 05''$, $B = 57^\circ\ 42'\ 43''$.
Ans. $C = 83^\circ\ 52'\ 12''$, $a = 249.98$ yds., $b = 340.09$.

2. Given $a = 471.5$, $A = 74^\circ\ 14'$, $B = 49^\circ\ 23'$.
Ans. $C = 56^\circ\ 23'$, $b = 371.9$, $c = 408.0$.

3. Given $c = 368$, $B = 79^\circ\ 46'\ 38''$, $C = 55^\circ\ 59'\ 58''$.
Ans. $A = 44^\circ\ 13'\ 24''$, $a = 309.6$, $b = 436.8$.

4. Given $c = 206.83$, $B = 106^\circ\ 10'$, $C = 50^\circ\ 25'\ 10''$.
Ans. $A = 23^\circ\ 24'\ 50''$, $a = 106.64$, $b = 257.74$.

5. Given $b = 159.06$, $B = 62^\circ\ 6'\ 51''$, $C = 53^\circ\ 27'\ 20''$.
Ans. $A = 64^\circ\ 25'\ 49''$, $a = 162.33$, $c = 144.61$.

6. Given $a = 46.706$, $c = 31.425$, $C = 10^\circ\ 14'\ 54''$.
Ans. $A = 15^\circ\ 19'\ 57''$, $B = 154^\circ\ 25'\ 9''$, $b = 76.268$,
$A' = 164^\circ\ 40'\ 3''$, $B' = 5^\circ\ 5'\ 3''$, $b' = 15.653$.

7. Given $a = 34.217$, $c = 45.90$, $A = 48^\circ\ 12'\ 20''$.
Ans. Impossible.

8. Given $a = 40$, $b = 50$, $A = 53^\circ\ 7'\ 48''$; find B.
Ans. $B = 90^\circ$.

9. Given $a = .042356$, $b = 0.051234$, $A = 55^\circ$.
Ans. $B = 82^\circ\ 14'\ 40''$, $C = 42^\circ\ 45'\ 20''$, $c = .035102$,
$B' = 97^\circ\ 45'\ 20''$, $C' = 27^\circ\ 14'\ 40''$, $c' = .023671$.

10. Given $a = .042356$, $b = 0.051234$, $A = 125^\circ$.
Ans. Impossible.

11. Given $a = 464.7$, $b = 289.3$, $C = 87^\circ\ 03'\ 48''$.
Ans. $A = 60^\circ\ 13'\ 38''$, $B = 32^\circ\ 42'\ 34''$, $c = 534.66$.

12. Given $b = 562$, $c = 320$, $A = 128^\circ\ 4'$.
Ans. $B = 33^\circ\ 34'\ 40''$, $C = 18^\circ\ 21'\ 20''$, $a = 800$.

13. Given $a = 424.096$, $b = 371.084$, $C = 21^\circ\ 47'\ 12''$.
Ans. $A = 98^\circ\ 12'\ 48''$, $B = 60^\circ$, $c = 159.036$.

14. Given $a = 31.0005$, $b = 15.1101$, $C = 10° 15$; find A and B.
Ans. $A = 160° 17' 14''$, $B = 9° 27' 46''$.

15. Given $a = .062387$, $c = .023475$, $B = 110° 32'$.
Ans. $A = 52° 10' 33''$, $C = 17° 17' 27''$, $b = .0739635$.

16. Given $a = 60$, $b = 80$, $c = 100$.
Ans. $A = 36° 52' 12''$, $B = 53° 7' 48''$. $C = 90°$.

17. Given $a = 126.2$, $b = 262.46$, $c = 208$.
Ans. $A = 28° 12' 04''$, $B = 100° 38' 30''$, $C = 51° 9' 26''$.

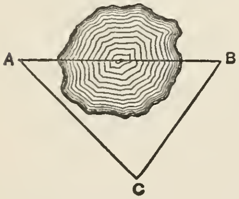
18. Given $a = 608.775$, $b = 1363.656$, $c = 949.689$.
Ans. $A = 22° 37' 12''$, $B = 120° 30' 36''$, $C = 36° 52' 14''$.

19. Given $a = 273.960$, $b = 198.632$, $c = 236.914$.
Ans. $A = 77° 24' 6''$, $B = 45° 2' 20''$, $C = 57° 33' 36''$.

20. Given $a = .8706$, $b = .0916$, $c = .7902$.
Ans. $A = 149° 49' 1''$, $B = 3° 1' 56''$, $C = 27° 9' 3''$.

Applications.

107. *To find a distance or an angle when it is impracticable to measure it directly.* Make the required distance or angle, one of the parts of a triangle, so selected, that three other parts, including a side, can be directly measured or otherwise determined. The required part, or parts, of the triangle may then be computed by the methods already explained.



1. To find the distance from a tree A to another, B, direct measurement being prevented by a pond or lake: measure the distances of a third point C from each of them, and the included angle ACB. Let

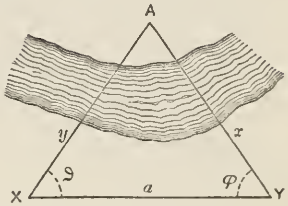
$$CB = 672 \text{ yds.},$$
$$CA = 588 \text{ yds.},$$
$$ACB = 55° 40'.$$

Compute AB. *Ans.* 592.967 yds.

2. To find the horizontal distance x of an inaccessible point A from a point Y in sight of it: select a point X, from which both A and Y can be seen; measure the horizontal distance $XY = a$, and the horizontal angles

$$YXA = \vartheta \quad \text{and} \quad AYX = \phi.$$

Then in the horizontal triangle XYA, compute $YA = x$.



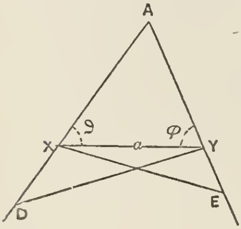
Example.

Given $a = 600$ yds., $\vartheta = 57° \; 35'$, $\phi = 64° \; 51'$. Compute $AY = x$ and $XA = y$. *Ans.* $x = 600.11$ yds., $y = 643.49$ yds.

3. Having no instruments for measuring angles, to find the horizontal distance $YA = x$: select D on AX prolonged, and E on the prolongation of AY. Measure $XY = a$, $YE = b$, $XD = c$, $XE = a'$ and $YD = a''$. Compute the angles DXY and XYE.

$$\vartheta = 180° - DXY, \quad \phi = 180° - XYE.$$

Compute x from a, ϑ and ϕ.

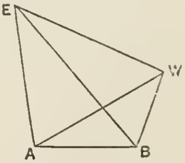


Example.

Given $a = 6$ ch., $a' = 6.42$ ch., $a'' = 7.94$, $b = 2$ ch., $c = 3$ ch. Compute $x = YA$ and $y = XA$.
Ans. $\vartheta = 59° \; 55' \; 10''$; $\phi = 87° \; 5' \; 40''$; $x = 9.54$ ch.; $y = 11.01$ ch.

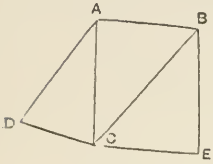
4. To find the horizontal distance between two inaccessible objects, E and W, the following measurements were made:

$$AB = 536 \text{ yds.},$$
$$BAW = 40° \; 16', \quad WBE = 71° \; 07',$$
$$WAE = 57° \; 40', \quad EBA = 42° \; 22'.$$



Compute EW. First by computing the triangle AWE, and then BWE. *Ans.* 939.62 yds.

5. To find the horizontal distance from an inaccessible point A to another, B, there being but one station, C, from which both A and B can be seen. Select two stations, D and E, so that C and A can be seen from D, and C and B, from E; then measure the parts given in the following example:



$CD = 200$ yds., $CE = 200$ yds.,

$ECB = 54° 30'$, $BEC = 88° 30'$,

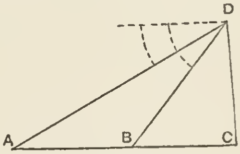
$BCA = 32° 25'$

$ACD = 86° 45'$, $CDA = 53° 30'$.

Compute AB.

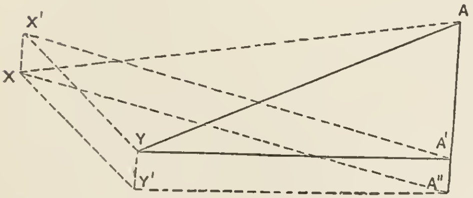
Suggestion. Compute CB from triangle BCE, and CA from ACD, then compute AB from ACB. *Ans.* 180.43 yds.

6. To find the distance between two inaccessible objects, A and B, which lie in a straight horizontal line from the bottom, C, of a tower 120 feet in height: the angles of these points below the horizontal were measured at the top, D, of the tower, and found to be $25° 30'$ for A, and $57°$ for B. Compute AB.



Ans. 173.656 ft.

7. To find the height of an inaccessible point A above a given point Y in sight of it, select a point X in sight of both A and Y, but not in range with them. Measure the horizontal distance from X to Y. At X measure the horizontal angle from Y to A, at Y the horizontal angle from X to A, and the vertical angle of A above the horizontal at Y. Compute as



in 2, the horizontal distance, YA', of A from Y, and in the right triangle $YA'A$, compute the required height $A'A$.

I. Given $X'Y = 780$ yds., $YX'A' = 41° 24'$,

$A'YX' = 96° 28'$, $A'YA = 7° 15'$.

Compute $A'A$. *Ans.* $YA' = 768.9$ yds., $A'A = 97.816$ yds.

II. Given in addition to the above,

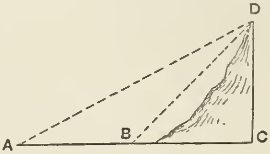
$$A''XA = 10° 43'.$$

Compute $A''A$. *Ans.* $XA'' = 1155.3$ yds., $A''A = 218.64$ yds.

If A were below the level of Y, the angle $A'YA$ would be measured below the horizontal line YA', and $A'A$ would be negative.

8. Required the height of a hill D above a horizontal plane AB, the distance from A to B being 975 yards, and the angles of elevation at A and B respectively 15° 36′ and 27° 29′.

Ans. 587.61 yds.



9. From the top of a house 42 feet in height the angle above the horizontal to the top of a spire is 14° 13′, and from the base of the house, on a level with that of the spire, it is 23° 19′. Find the height of the spire. *Ans.* 101.891 ft.

10. From the top of a mountain three miles high, the angle of depression to a line tangent to a vertical circle of the earth's surface is found to be 2° 13′ 30″. Compute the radius of the earth.

Ans. 3976 miles.

Suggestion. Denoting the observed angle by ϕ, the angle between the tangent and the chord to the base of the mountain is $\frac{1}{2}\phi$, which enables us to compute the tangent line in the triangle formed by it with the chord and the mountain's height. The radius is equal to the tangent line divided by tan ϕ.

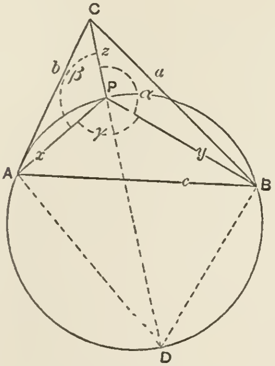
Otherwise, denoting the given height by a, we have

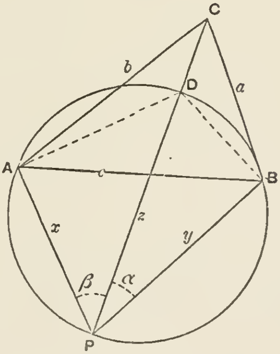
$$a = R \sec \phi - R = R \frac{1 - \cos \phi}{\cos \phi} = \frac{2R \sin^2 \frac{\phi}{2}}{\cos \phi};$$

$$\therefore \quad R = \frac{a \cos \phi}{2 \sin^2 \frac{\phi}{2}}.$$

Pothenot or Snellius.

108. Three-point Problem.—In the triangle ABC, let the angles and the sides be known from measurement and computation. From any point, as P, the horizontal angles $BPC = \alpha$, $APC = \beta$, and $BPA = \gamma$, are measured. γ serves as a check. When $\gamma = \alpha + \beta$, P is without; and when $\gamma = 360° - (\alpha + \beta)$, P is within the triangle.





The problem is to determine the distances AP, CP, and BP. Represent them, respectively, by x, z and y. Let APB represent the circumference through A, B and P, and let D be the second point where CP, or CP produced, intersects the circumference. Draw the chords AD and BD. The angle $BAD = BPD$, because both are inscribed in the same segment. $BPD = \alpha$, or $180° - \alpha$, according as P is without or within the triangle. Hence $BAD = \alpha$, or $180° - \alpha$. Similarly, $ABD = APD = \beta$, or $180° - \beta$.

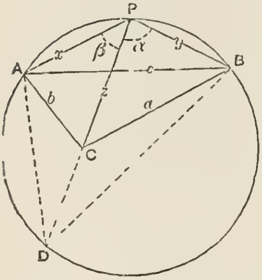
In the triangle ADB, the side c, and the angles BAD and ABD, are known. Compute the sides AD and BD.

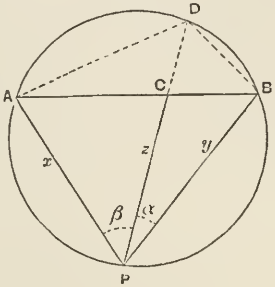
In the triangle ADC, the angle $DAC = BAC \mp BAD$, and the sides AD and AC, are known. Compute the angle ACD.

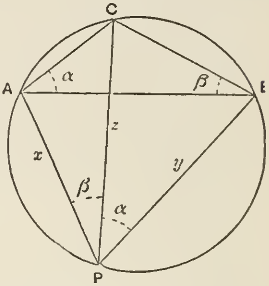
In the triangle ACP, the angles ACP and APC, and the side AC, are known. Compute the sides $AP = x$, and $CP = z$.

In the triangle BPC, the angles BPC and $BCP = C - ACP$, and the side CB are known. Compute the side $BP = y$.

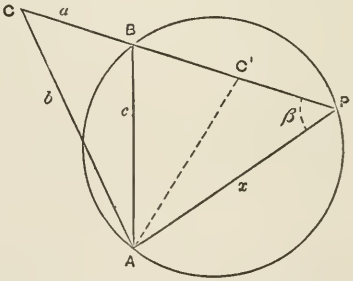
The following figures illustrate different cases that may occur.







In the last case, the angle $BAC = \alpha$, and $ABC = \beta$; P is on the circumference through A, B and C, and the problem is indeterminate.



If one of the measured angles, as α, is zero, in the triangle ACP, the angles ACP and APC, and the side AC, are known. Compute the sides $AP = x$, and $CP = z$; then $BP = z \mp a$.

If both α and β are zero, P is at an infinite distance, and the solution is impossible, or the points A, B, C and P are on the same right line, and the problem is indeterminate.

Geometric Construction of P. When P is without the triangle, draw AD, on the side of AB opposite from P, making the angle $BAD = \alpha$; and similarly, draw BD, making the angle $ABD = \beta$.

P will be at the intersection of the right line CD with the circumference ADB.

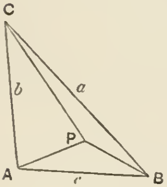
When P is within the triangle, construct the angle $BAD = 180° - \alpha$, and $ABD = 180° - \beta$.

Examples.

1. Given $a = 76$, $b = 63$, $c = 44$, $APC = 89° \ 15'$, $CPB = 130° \ 45'$; compute $AP = x$, $BP = y$, and $CP = z$.

Ans. $A = 88° \ 40'$, $B = 55° \ 58' \ 4''$;

$ACP = 21° \ 57' \ 17''$, $x = 23.556$,
$PCB = 13° \ 24' \ 38''$, $y = 23.267$,
$ABP = 20° \ 6' \ 43''$: $z = 58.74$.



2. Given $a = 538.496$, $APC = 23° \ 9'$,
$b = 610.00$, $APB = 14° \ 6'$;
$c = 426.76$;
compute $AP = x$, $BP = y$, and $CP = z$. (2)

Ans. $A = 59° \ 30'$, $B = 77° \ 26'$, $C = 43° \ 4'$;

$ABP = 57° \ 26' \ 46''$, $x = 1476.6$,
$PBC = 19° \ 59' \ 14''$, $y = 1661.7$,
$ACP = 108° \ 27' \ 14''$; $z = 1170.2$.

3. Given a, b, c, same as in Ex. 2. $AP'C = 23° \ 9'$, $AP'B = 14° \ 6'$. compute $AP' = x'$, $BP' = y'$, $CP' = z'$. (3)

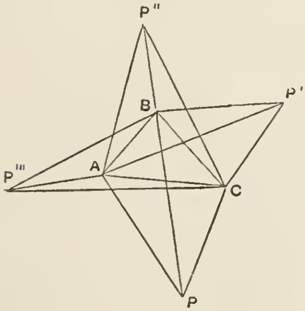
Ans. A, B and C same as above;

$P'AC = 29° \ 7' \ 34''$, $x' = 1227.3$,
$P'AB = 30° \ 22' \ 26''$, $y' = 885.78$,
$P'BC = 58° \ 05' \ 34''$; $z' = 755.23$.

4. Given a, b, c, as in Ex. 2. $AP''C = 23° \ 9'$, $AP''B = 14° \ 6'$. Compute $AP'' = x''$, $BP'' = y''$, $CP'' = z''$.

Ans. A, B and C as in Ex. 2;

$CBP'' = 153° \ 12' \ 40''$, $x'' = 1354.5$,
$ABP'' = 129° \ 21' \ 20''$, $y'' = 1043.1$,
$CAP'' = 96° \ 2' \ 40''$; $z'' = 1543.0$.



5. Given a, b, c, as in Ex. 2. $AP'''C = 23° 9'$, $AP'''B = 14° 6'$. Compute $AP''' = x'''$, $BP''' = y'''$, $CP''' = z'''$.

Ans. A, B and C as in Ex. 2.

$$\begin{aligned} P'''AC &= 145° 02' 32'', & x''' &= 317.5; \\ P'''AB &= 155° 27' 30'', & y''' &= 727.62; \\ P'''BC &= 87° 52' 32'': & z''' &= 889.02. \end{aligned}$$

6. Given $a = 76$, $b = 63$, $c = 44$; $\alpha = 130° 45'$, $\beta = 89° 15'$; compute x, y and z. *Ans.* $x = 23.556$, $y = 23.267$, $z = 58.74$, or $x = 60.709$, $y = 63.295$, $z = 17.648$.

7. Given $a = 400$, $b = 800$, $c = 600$; $\alpha = 22° 30'$, $\gamma = 33° 45'$; compute x, y and z. *Ans.* $x = 706.43$, $y = 1041.2$, $z = 935.6$.

PART II.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

RIGHT TRIHEDRALS AND SPHERICAL TRIANGLES.

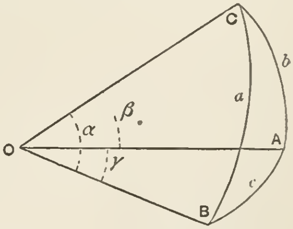
109. In every spherical triangle six parts are considered: three *sides* and three *angles*.

In order to solve a spherical triangle, it is necessary to compute the unknown parts of the trihedral formed by the planes of the three sides of the triangle.

In the solution of trihedrals six parts are considered: three *face-angles* and three *dihedrals*.

110. Let ABC be any spherical triangle on a sphere whose radius is R. Denote the angles of the triangle by A, B and C, and their opposite sides by a, b and c respectively.

Let O be the centre of the sphere. Draw the radii OA, OB and OC. They are the edges of the *trihedral* formed by the planes of a, b and c.



It has been shown in Geometry that the angle between two arcs of great circles at their point of intersection is equal to the angle between their planes. Hence:

The three angles of the triangle are respectively equal to their corresponding dihedrals, which are also denoted by A, B and C, respectively. Denote the *face-angles* BOC, COA and AOB by α, β and γ, respectively. Then

$$\alpha = \frac{a}{R}, \qquad \beta = \frac{b}{R}, \qquad \gamma = \frac{c}{R}, \quad . \quad . \quad (115)$$

in which α, β and γ are in radians.

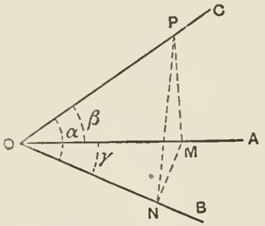
Hence: *The radian measure of each face-angle of the trihedral is equal to the corresponding side of the triangle divided by the radius of the sphere.*

111. If the three planes of the face-angles of any trihedral be produced, they will form eight trihedrals, and their intersections with the surface of the sphere will form eight corresponding triangles.

Unless it is otherwise expressed, those trihedrals only in which each part is less than 180° will be considered. Each side of any corresponding triangle will, therefore, be less than a semi-circumference Each part of a trihedral is considered as positive.

Right Trihedrals.

112. Relations of Parts. A right trihedral has one of its dihedrals a right angle.



Let OA, OB and OC be the edges, and O the vertex of any right trihedral. Let $A = 90°$.

Through any point P of OC pass a plane perpendicular to OB, cutting OA at M, and OB at N. Draw MN, MP and NP.

By construction, NM and NP are perpendicular to OB, and $MNP = B$. Also, MP, being the intersection of two planes perpendicular to that of AOB, is perpendicular to OA. Then

$$\cos\alpha = \frac{ON}{OP} = \frac{OM}{OP} \cdot \cos\gamma$$

or

$$\cos\alpha = \cos\beta\cos\gamma. \quad . \quad . \quad (116)$$

Also, $$\sin\beta = \frac{MP}{OP} = \frac{NP}{OP}\sin B,$$

or $$\sin\beta = \sin\alpha\sin B; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (117)$$

from which, by analogy, write,

$$\sin\gamma = \sin\alpha\sin C. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (118)$$

From the figure we have

$$\cos B = \frac{NM}{NP} = \frac{OM\sin\gamma}{OP\sin\alpha} = \cos\beta\,\frac{\sin\gamma}{\sin\alpha}.$$

But from (118) we have $\dfrac{\sin\gamma}{\sin\alpha} = \sin C$.

Hence, $$\cos B = \cos\beta\sin C; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (119)$$

from which, by analogy, write,

$$\cos C = \cos\gamma\sin B. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (120)$$

In (116) eliminate $\cos\beta$ by (119), and $\cos\gamma$ by (120), giving

$$\cos\alpha = \frac{\cos B}{\sin C}\cdot\frac{\cos C}{\sin B} = \frac{\cos B}{\sin B}\cdot\frac{\cos C}{\sin C},$$

or $$\cos\alpha = \cot B\cot C. \quad . \; . \; . \; . \; . \; . \; . \quad (121)$$

In (117) eliminate $\sin\alpha$ by (118), and $\sin B$ by (120), giving, after reduction,

$$\sin\beta = \tan\gamma\cot C; \quad . \; . \; . \; . \; . \; . \; . \quad (122)$$

from which, by analogy, write,

$$\sin\gamma = \tan\beta\cot B. \quad . \; . \; . \; . \; . \; . \quad (123)$$

In (119) eliminate $\cos\beta$ by (116), $\sin C$ by (118), and reduce, giving

$$\cos B = \cot\alpha\ \tan\gamma; \quad \ldots \ldots \quad (124)$$

from which, by analogy, write,

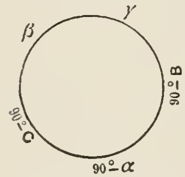
$$\cos C = \cot\alpha\ \tan\beta. \quad \ldots \ldots \quad (125)$$

These formulas, expressing relations between functions of the parts of any right trihedral, are arranged for reference in

TABLE E.

$\cos\alpha = \cos\beta\cos\gamma.$	$\cos\alpha = \cot B\cot C.$
$\sin\beta = \sin\alpha\sin B.$	$\sin\beta = \tan\gamma\cot C.$
$\sin\gamma = \sin\alpha\sin C.$	$\sin\gamma = \tan\beta\cot B.$
$\cos B = \cos\beta\sin C.$	$\cos B = \cot\alpha\tan\gamma.$
$\cos C = \cos\gamma\sin B.$	$\cos C = \cot\alpha\tan\beta.$

113. Napier's Rule. In order to deduce a rule by which these equations may be formed, Napier arbitrarily substituted for the parts B, C and α, their complements, giving for the parts considered β, γ, $90^\circ - \alpha$, $90^\circ - B$, $90^\circ - C$, which are called *Napierian parts* in right trihedrals. They are considered consecutive in the order indicated in the figure, which is the order of the corresponding parts of the trihedral, omitting from consideration the right angle A. Each part considered as a *middle part* has two *adjacent* parts and two *opposite* parts.



The formulas in Table E may be expressed in *Napierian parts*, thus—

$$\sin(90^\circ - \alpha) = \cos\beta\cos\gamma,$$
$$\sin(90^\circ - \alpha) = \tan(90^\circ - B)\tan(90^\circ - C);$$

$$\sin\beta = \cos(90^\circ - \alpha)\cos(90^\circ - B),$$
$$\sin\beta = \tan\gamma\ \tan(90^\circ - C);$$

$$\sin \gamma = \cos (90° - \alpha) \cos (90° - C)$$
$$\sin \gamma = \tan \beta \tan (90° - B);$$

$$\sin (90° - B) = \cos \beta \cos (90° - C),$$
$$\sin (90° - B) = \tan (90° - \alpha) \tan \gamma;$$

$$\sin (90° - C) = \cos \gamma \cos (90° - B),$$
$$\sin (90° - C) = \tan (90° - \alpha) \tan \beta.$$

Hence the rule:

The sine of the middle Napierian part is equal to the product of the cosines of its opposite parts, or to the product of the tangents of its adjacent parts.

To find the relations of B, C and γ by this rule. Pass to the Napierian parts $90° - B$, $90° - C$, and γ, of which $90° - C$ is the middle part, $90° - B$ and γ being its opposite parts.

$$\therefore \sin (90° - C) = \cos (90° - B) \cos \gamma,$$

or
$$\cos C = \cos \gamma \sin B,$$

which is (120).

114. Table E contains but three independent equations, for Equations (121) to (125) can be formed from Equations (116) to (120), and depend on them.

Of these take any three, as (116), (117) and (118).

No one of these can be dependent on the other two, since the parts are different, and since the elimination of a common part from the two cannot produce an equation with exactly the parts of the one.

To show that (119) is dependent on these three, eliminate α and γ from the group. Thus—

From (117), $$\sin \alpha = \frac{\sin \beta}{\sin B}.$$

From (118), $$\sin \gamma = \sin \alpha \sin C = \frac{\sin \beta}{\sin B} \sin C.$$

These values in (116), after squaring both members and eliminating cosines by (1), give

$$1 - \frac{\sin^2 \beta}{\sin^2 B} = (1 - \sin^2 \beta)\left(1 - \frac{\sin^2 \beta}{\sin^2 B} \sin^2 C\right),$$

or $$-\frac{\sin^2 \beta}{\sin^2 B} = -\sin^2 \beta - (1 - \sin^2 \beta) \frac{\sin^2 \beta}{\sin^2 B} \sin^2 C.$$

Divide both members by the first member and transpose, giving

$$1 - \sin^2 B = (1 - \sin^2 \beta) \sin^2 C,$$

or $$\cos^2 B = \cos^2 \beta \sin^2 C,$$

which is (119) with both members squared.

To show the dependence of (120) on (116), (117) and (118), eliminate α and β in like manner, giving, after reduction,

$$\cos^2 C = \cos^2 \gamma \sin^2 B,$$

which is (120) with both members squared.

Therefore, Table E contains but three independent equations.

115. Two parts of any trihedral are of the *same species* if they are both acute or both obtuse; of *different species* if one is acute and the other obtuse.

PRINCIPLES OF RIGHT TRIHEDRALS.

1st. *Each oblique dihedral is of the same species as its opposite face-angle, and if one is* 90° *the other is* 90°.

From (119), $$\sin C = \frac{\cos B}{\cos \beta}.$$

sin C is always positive; hence, cos B and cos β have like signs, and B is of the same species as β.

Since $\sin C$ cannot be 0, or ∞, for any possible trihedral, $B = 90^\circ$ gives $\beta = 90^\circ$, and *vice versa.*

2d. *If the face-angle opposite to the right dihedral is acute, its adjacent dihedrals are oblique and of the same species; if it is obtuse, they are oblique but of different species; if it is a right angle, one of them is also a right angle, and the other may be an acute, obtuse or a right angle.*

From (121) we have

$$\cos \alpha = \cot B \cot C.$$

If $\alpha < 90^\circ$, $\cos \alpha$ is positive. $\therefore$ $\cot B$ and $\cot C$ have like signs, and B is of the same species as C.

If $\alpha > 90^\circ$, $\cos \alpha$ is negative. $\therefore \cot B$ and $\cot C$ have contrary signs, and B is of different species from C.

If $\alpha = 90^\circ$, $\cos \alpha = 0$, giving $0 = \cot B \cot C$, which may be satisfied by $B = 90^\circ$ independent of C, or by $C = 90^\circ$ independent of B.

3d. *If the face-angle opposite to the right dihedral is acute, the face-angles adjacent to the right dihedral are oblique and of the same species; if it is obtuse, they are oblique and of different species; if it is a right angle, one of them is also a right angle, and the other may be an acute, obtuse or a right angle.*

This follows from the 1st and 2d principles, and may be proven directly from (116).

4th. *The face-angles adjacent to an oblique dihedral are of the same species if the dihedral is acute; of different species if it is obtuse.*

From (124), $\cos B = \cot \alpha \tan \gamma.$

If $B < 90^\circ$, α and γ are of the same species.

If $B > 90^\circ$, α and γ are of different species.

This conclusion fails if $\alpha = 90^\circ$, or $\gamma = 90^\circ$.

5th. *An oblique dihedral, if not equal to its opposite face-angle, is nearer than it, to* 90°

From (117) we have $\sin B = \dfrac{\sin \beta}{\sin \alpha}.$

If $\sin \alpha = 1$, we have $\sin B = \sin \beta$; or, by the 1st principle, $B = \beta$. Otherwise, $\sin \alpha < 1$, and $\sin B > \sin \beta$, giving B nearer 90° than β.

6th. *The face-angle opposite to the right dihedral is nearer* 90° *than a face-angle opposite to an oblique dihedral.*

From (117), $$\sin \alpha = \frac{\sin \beta}{\sin B}.$$

If B is oblique, $\sin B < 1$, $\sin \alpha > \sin \beta$. and α is nearer 90° than β.

7th. *The difference of the two oblique dihedrals is less than* 90°, *and their sum is greater than* 90°, *but less than* 270°.

1°. From Geometry, we have $\alpha > \beta - \gamma$.

Passing to the polar trihedral*, we have

$$180^\circ - A' > C' - B'.$$

When $A' = 90^\circ$, $$90^\circ > C' - B'.$$

Similarly, we obtain $$90^\circ > B' - C'.$$

2°. From Geometry, we have $A + B + C > 180^\circ$

When $A = 90^\circ$, $$B + C > 90^\circ.$$

3°. From Geometry, we have $\beta + \gamma > \alpha$.

Passing to the polar trihedral, we have

$$\begin{aligned} 360^\circ - (B' + C') &> 180^\circ - A'; \\ -(B' + C') &> -180^\circ - A'; \\ B' + C' &< 180^\circ + A'. \end{aligned}$$

When $A' = 90^\circ$, $$B' + C' < 270^\circ.$$

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* Two trihedrals are *polar* when their corresponding spherical triangles on the same sphere are *polar*. [Geometry.]

SOLUTION OF RIGHT TRIHEDRALS.

116. Since only three of the formulas in Table E are independent (§ 114), and they involve the five quantities α, β, γ, B and C, at least two of them must be given in order to compute the others.

There are six cases. There may be given in addition to the right dihedral—

I. The two face-angles about the right dihedral.

II. The face-angle opposite to the right dihedral and another face-angle.

III. The face-angle opposite to the right dihedral and one oblique dihedral.

IV. One of the face-angles about the right dihedral and the adjacent oblique dihedral.

V. One of the face-angles about the right dihedral and the opposite oblique dihedral.

VI. The two oblique dihedrals.

Unless otherwise indicated, the following *general method* should be followed:

See that the given parts are positive, each less than 180°, and consistent with the principles of § 115.

Find by Napier's rule an equation containing the two given and one of the required parts.

Solve the equation with reference to a function of the required part, and compute the part.

Proceed in this manner with each required part.

The three computed parts may then be checked by the direct relation between them.

117. CASE I. *Given the two face-angles about the right dihedral; or β and γ.* The formulas are:

$$\cos \alpha = \cos \beta \cos \gamma, \qquad \cot B = \frac{\sin \gamma}{\tan \beta}, \qquad \cot C = \frac{\sin \beta}{\tan \gamma};$$

check,
$$\cos \alpha = \cot B \cot C.$$

If $\beta = 90°$, we have $\alpha = 90°$, $B = 90°$, and $C = \gamma$.

If α is near 0°, it cannot be accurately computed from its cosine, but it may be accurately computed by (124) from γ and B, or by (125) from β and C

Examples.

1. Given $\beta = 155° \; 27' \; 54''$. $\gamma = 29° \; 46' \; 08''$, solve the trihedral.

Solution α.		*P*	
$\log \cos \beta$	$9.9589\hat{6}$	a. c. $\log \tan \beta$	0.34059
$\log \cos \gamma$	$0.9385\hat{3}$	$\log \sin \gamma$	9.69592
$\log \cos \alpha$	9.89744	$\log \cot B$	0.03651
$37° \; 50' \; 47''$		$42° \; 35' \; 40''$	
$\alpha = 142° \; 9' \; 13''$		$B = 137° \; 24' \; 20''$	

C		Check.	
$\log \sin \beta$	9.61831	$\log \cot B$	0.03651
a. c. $\log \tan \gamma$	$0.2426\hat{1}$	$\log \cot C$	$9.8609\hat{2}$
$\log \cot C$	$9.8609\hat{2}$	$\log \cos \alpha$	$9.8974\hat{3}$
$54° \; 01' \; 16''$			
$C = 54° \; 01' \; 16''$			

2. Given $\beta = 16°$, $\gamma = 116°$, solve the trihedral.

Ans. $\alpha = 114° \; 55' \; 20''$, $B = 17° \; 41' \; 40''$, $C = 97° \; 39' \; 24''$

118. Case II. *Given the face-angle opposite to the right dihedral and another face-angle: or* α *and* β. The formulas are:

$$\cos \gamma = \frac{\cos \alpha}{\cos \beta}, \qquad \sin B = \frac{\sin \beta}{\sin \alpha}, \qquad \cos C = \cot \alpha \tan \beta;$$

check,
$$\cos C = \cos \gamma \sin B.$$

If $\alpha = 90°$, and β is oblique. we have

$$\gamma = 90°, \; B = \beta, \text{ and } C = 90°.$$

If $\alpha = 90°$ and $\beta = 90°$, we have

$$\cos \gamma = \frac{0}{0}, \; \sin B = 1, \; \cos C = 0 \times \infty, \text{ and } \cos C = \cos \gamma;$$

or
$$B = 90°, \qquad \gamma = C \frac{0}{0}.$$

Examples.

1. Given $\alpha = 140^\circ$, $\beta = 20^\circ$, solve the trihedral.

	γ.		B.
log cos α	$9.8842\hat{5}$	a. c. log sin α	$0.1919\hat{3}$
a. c. log cos β	$0.0270\hat{1}$	log sin β	9.53405
log cos γ	9 91127	log sin B	$9.7259\hat{8}$
	$35^\circ\ 23'\ 30''$		$32^\circ\ 08'\ 48''$
$\gamma = 144^\circ\ 36'\ 30''$		$B = 32^\circ\ 08'\ 48''$ [1st principle].	

	C.		Check.
log cot α	$0.0761\hat{8}$	log cos γ	9.91127
log tan β	$9.5610\hat{6}$	log sin B	$9.7259\hat{8}$
log cos C	9.63725	log cos C	$9.6372\hat{5}$
	$64^\circ\ 17'\ 37''$		
$C = 115^\circ\ 42'\ 23''$			

2. Given $\alpha = 40^\circ$, $\beta = 20^\circ$, solve the trihedral.
Ans. $\gamma = 35^\circ\ 23'\ 30''$, $B = 32^\circ\ 08'\ 48''$, $C = 64^\circ\ 17'\ 37''$.

3. Given $\alpha = 40^\circ$, $\beta = 60^\circ$, solve the trihedral.
Ans. Impossible [6th principle].

4. Given $\alpha = 40^\circ$, $\beta = 160^\circ$, solve the trihedral.
Ans. $\gamma = 144^\circ\ 36'\ 30''$, $B = 147^\circ\ 51'\ 12''$, $C = 115^\circ\ 42'\ 23''$..

5. Given $\alpha = 140^\circ$, $\beta = 160^\circ$, solve the trihedral.
Ans. $\gamma = 35^\circ\ 23'\ 30''$, $B = 147^\circ\ 51'\ 12''$, $C = 64^\circ\ 17'\ 37''$.

6. Given $\alpha = 160^\circ$, $\beta = 140^\circ$, solve the trihedral.
Ans. Impossible [6th principle].

7. Given $\alpha = 90^\circ$, $\beta = 20^\circ$, solve the trihedral.
Ans. $\gamma = 90^\circ$, $B = 20^\circ$, $C = 90^\circ$.

8. Given $\alpha = 160°$, $\beta = 90°$, solve the trihedral.

Ans. Impossible.

9. Given $\alpha = 90°$, $\beta = 90°$, solve the trihedral.

Ans. $\gamma = \frac{0}{0}$, $B = 90°$, $C = \frac{0}{0}$.

10. Given $\alpha = 105° \; 17' \; 29''$, $\beta = 38° \; 47' \; 11''$, find $C = 102° \; 41' \; 33''$.

119. If γ and C are near 0°, and B near 90°, the above formulas will not give accurate angles.

They may be written thus—

$$\cos \gamma = \frac{\cos \alpha}{\cos \beta},$$

$$\sin B = \frac{\sin \beta}{\sin \alpha},$$

$$\cos C = \cot \alpha \tan \beta = \frac{\cos \alpha \sin \beta}{\sin \alpha \cos \beta}.$$

Subtract both members of each from unity, add both members to unity, divide and reduce, giving

$$\frac{1 - \cos \gamma}{1 + \cos \gamma} = \frac{\cos \beta - \cos \alpha}{\cos \beta + \cos \alpha}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

$$\frac{1 - \sin B}{1 + \sin B} \text{ or } \frac{1 - \cos (90° - B)}{1 + \cos (90° - B)} = \frac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta}, \quad . \quad (b$$

$$\frac{1 - \cos C}{1 + \cos C} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta}. \quad . \; . \; . \; . \; . \quad (c)$$

Reduce the second member of (*a*) by (69); then, by (48), we have

$$\tan \tfrac{1}{2}\gamma = \sqrt{\frac{1 - \cos \gamma}{1 + \cos \gamma}} = \sqrt{\tan \tfrac{1}{2}(\alpha + \beta) \tan \tfrac{1}{2}(\alpha - \beta)}. \quad (126)$$

Similarly reducing (b) by (66), we have

$$\tan \tfrac{1}{2}(90° - B) = \pm \sqrt{\frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}}; \quad . \quad . \quad . \quad . \quad (127)$$

and reducing (c) by (29) and (30), we have

$$\tan \tfrac{1}{2}C = \sqrt{\frac{\sin (\alpha - \beta)}{\sin (\alpha + \beta)}}; \quad . \quad . \quad . \quad . \quad . \quad (128)$$

formulas which give accurate results.

Examples.

1. Given $\alpha = 142° \ 20'$, $\beta = 142° \ 21'$, solve the trihedral.
Ans. $\gamma = 1° \ 12' \ 50''$, $B = 91° \ 34' \ 23''$, $C = 1° \ 59' \ 13''$.

2. Given $\alpha = 85°$, $\beta = 84° \ 59' \ 45''$, solve the trihedral.
Ans. $\gamma = 3° \ 18' \ 6''$, $B = 89° \ 42' \ 38''$, $C = 3° \ 18' \ 50''$.

120. CASE III. *Given the face-angle opposite to the right dihedral and one oblique dihedral; or α and B.*

The formulas, by the general method § 116, are

$$\sin \beta = \sin \alpha \sin B, \quad \tan \gamma = \frac{\cos B}{\cot \alpha}, \quad \cot C = \frac{\cos \alpha}{\cot B};$$

check, $\sin \beta = \tan \gamma \cot C.$

The 1st principle of § 115 will decide which value of β is admissible. If β is found to be near 90°, it should be computed from B and γ or from α and C.

If $\alpha = 90°$, we have

$$\beta = B, \gamma = 90°, \text{ and } C = 90°.$$

If $B = 90°$, it requires, by (121), $\alpha = 90$.

Examples.

1. Given $\alpha = 110° \ 46' \ 20''$, $B = 80° \ 10' \ 30''$, solve the trihedral.

Ans. $\beta = 67° \ 06' \ 48''$, $\gamma = 155° \ 46' \ 46''$, $C = 153° \ 58' \ 24''$.

2. Given $\alpha = 110° \ 46' \ 20''$, $B = 99° \ 49' \ 30''$, solve the trihedral.

Ans. $\beta = 112° \ 53' \ 12''$, $\gamma = 24° \ 13' \ 14''$, $C = 26° \ 01' \ 36''$.

3. Given $\alpha = 69° \ 13' \ 40''$, $B = 99° \ 49' \ 30''$, solve the trihedral.

Ans. $\beta = 112° \ 53' \ 12''$, $\gamma = 155° \ 46' \ 46''$, $C = 153° \ 58' \ 24''$.

121. CASE IV. *Given one of the face-angles about the right dihedral and the adjacent oblique dihedral; or β and C.*

The formulas, by the general method, are

$$\cot \alpha = \frac{\cos C}{\tan \beta}, \qquad \tan \gamma = \frac{\sin \beta}{\cot C}, \qquad \cos B = \cos \beta \sin C;$$

check,
$$\cos B = \cot \alpha \ \tan \gamma.$$

If B is found to be near 0°, it should be computed from β and γ, or from α and C.

If β is 90° and C oblique, we have

$$\alpha = 90°, \quad \gamma = C, \quad B = 90°.$$

If β is oblique and C is 90°, we have

$$\alpha = 90°, \quad \gamma = 90°, \quad B = \beta.$$

If $\beta = C = 90°$, we have $\alpha = \gamma = B = 90°$.

Examples.

1. Given $\beta = 155° \ 46' \ 43'$, $C = 80° \ 10' \ 30''$, solve the trihedral.

Ans. $\alpha = 110° \ 46' \ 17''$, $\gamma = 67° \ 06' \ 53''$, $B = 153° \ 58' \ 20''$.

2. Given $\beta = 155° \ 46' \ 43''$, $C = 99° \ 49' \ 30''$, solve the trihedral.

Ans. $\alpha = 69° \ 13' \ 43''$, $\gamma = 112° \ 53' \ 07''$, $B = 153° \ 57' \ 20''$.

3. Given $\gamma = 73° \ 41' \ 35''$, $B = 99° \ 17' \ 33''$, solve the trihedral.

Ans. $\alpha = 92° \ 42' \ 17''$, $\beta = 99° \ 10' \ 30''$, $C = 73° \ 54' \ 47''$.

122. CASE V. *Given one face-angle about the right dihedral and the opposite oblique dihedral; or β and B.*

The formulas for the general method are

$$\sin\alpha = \frac{\sin\beta}{\sin B}, \qquad \sin\gamma = \tan\beta\cot B = \frac{\tan\beta}{\tan B},$$

$$\sin C = \frac{\cos B}{\cos\beta}, \qquad \text{and,} \qquad \text{check,} \qquad \sin\gamma = \sin\alpha\sin C.$$

If $\sin\alpha = 1$, $\alpha = 90°$, which requires $\beta = B$, and $\gamma = C = 90°$

If $\sin\alpha < 1$, then $\alpha < 90°$, and $\alpha' = 180° - \alpha$, are consistent with the given parts, giving two solutions.

The principles of § 115 determine which values of α, γ and C form a triangle with the given values of β and B.

Examples.

1. Given $\beta = 115° \ 20'$, $B = 91° \ 1' \ 47''$, solve the trihedral.

Computation:

	α		γ
log sin β	9.95609	log tan β	$0.3247\hat{6}$
a. c. log sin B	0.00007	log cot B	$8.2546\hat{4}$
log sin α	9.95616	log sin γ	8.57941

$64^\circ\ 41'\ 10''$ $\qquad$ $2^\circ\ 10'\ 33''$

$\alpha = 64^\circ\ 41'\ 10''$ $\qquad$ $\gamma = 177^\circ\ 49'\ 27''$ [4th principle]

$\alpha' = 115^\circ\ 18'\ 50''$ $\qquad$ $\gamma' = 2^\circ\ 10'\ 33''$ [3d principle]

C		Check.	
a. c. log cos β	$0.3686\hat{7}$	log sin α	9.95616
log cos B	$8.2545\hat{7}$	log sin C	8.62325
log sin C	8.62325	log sin γ	8.57941

$2^\circ\ 24'\ 26''$

$C = 177^\circ\ 35'\ 34''$ [2d principle]

$C' = 2^\circ\ 24'\ 26''$.

2. Given $\beta = 91^\circ\ 1'\ 47''$, $B = 115^\circ\ 20'$, solve the trihedral.
Ans. Impossible [5th principle].

3. Given $\gamma = 112^\circ$, $C = 100^\circ$, solve the trihedral.
Ans. $\alpha = 70^\circ\ 18'\ 11''$, $\beta = 154^\circ\ 7'\ 27''$, $B = 152^\circ\ 23'\ 01''$;
$\alpha' = 109^\circ\ 41'\ 49''$, $\beta' = 25^\circ\ 52'\ 33''$, $B' = 27^\circ\ 36'\ 59''$.

4. Given $\gamma = 68^\circ$, $C = 100^\circ$, solve the trihedral.
Ans. Impossible [1st principle].

5. Given $\gamma = 86^\circ$, $C = 100^\circ$, solve the trihedral.
Ans. Impossible [1st and 5th principles].

6. Given $\gamma = 86^\circ$, $C = 80^\circ$, solve the trihedral.
Ans. Impossible [5th principle]

7. Given $\gamma = 68^\circ$, $C = 80^\circ$, solve the trihedral.
Ans. $\alpha = 70^\circ\ 18'\ 11''$, $\beta = 25^\circ\ 52'\ 33''$, $B = 27^\circ\ 36'\ 59''$;
$\alpha' = 109^\circ\ 41'\ 49''$, $\beta' = 154\ 07'\ 27''$, $B' = 152^\circ\ 23'\ 01''$.

8. Given $\beta = 51^\circ\ 30'$, $B = 58^\circ\ 35'$, solve the trihedral.
Ans. $\alpha = 66^\circ\ 29'\ 54''$, $\gamma = 50^\circ\ 09'\ 51''$, $C = 56^\circ\ 51'\ 38''$;
$\alpha' = 113^\circ\ 30'\ 06''$, $\gamma' = 129^\circ\ 50'\ 09''$, $C' = 123^\circ\ 08'\ 22''$.

123. If any required part is near 90°, it cannot be accurately computed by the general method.

To deduce formulas applicable in such cases, write the foregoing formulas,

$$\sin\alpha = \frac{\sin\beta}{\sin B}, \qquad \sin\gamma = \frac{\cos B\sin\beta}{\sin B\cos\beta}, \qquad \sin C = \frac{\cos B}{\cos\beta}.$$

Subtract both members of each formula from unity, add both to unity, divide and reduce, giving

$$\frac{1-\sin\alpha}{1+\sin\alpha} \text{ or } \frac{1-\cos(90^\circ-\alpha)}{1+\cos(90^\circ-\alpha)} = \frac{\sin B-\sin\beta}{\sin B+\sin\beta}, \quad . \quad (a)$$

$$\frac{1-\cos(90^\circ-\gamma)}{1+\cos(90^\circ-\gamma)} = \frac{\sin B\cos\beta-\cos B\sin\beta}{\sin B\cos\beta+\cos B\sin\beta}, \quad . \quad (b)$$

$$\frac{1-\cos(90^\circ-C)}{1+\cos(90^\circ-C)} = \frac{\cos\beta-\cos B}{\cos\beta+\cos B}. \quad . \; . \; . \; . \; . \; . \quad (c)$$

Reduce (*a*) by (66), (*b*) by (29) and (30), and (*c*) by (69), giving, from (48),

$$\tan\tfrac{1}{2}(90^\circ-\alpha) = \pm\sqrt{\frac{1-\cos(90^\circ-\alpha)}{1+\cos(90^\circ-\alpha)}} = \pm\sqrt{\frac{\tan\frac{1}{2}(B-\beta)}{\tan\frac{1}{2}(B+\beta)}}, \quad (129)$$

$$\tan\tfrac{1}{2}(90^\circ-\gamma) = \pm\sqrt{\frac{\sin(B-\beta)}{\sin(B+\beta)}}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (130)$$

$$\tan\tfrac{1}{2}(90^\circ-C) = \pm\sqrt{\tan\tfrac{1}{2}(B-\beta)\tan\tfrac{1}{2}(B+\beta)}. \quad . \; . \; . \; . \; . \quad (131)$$

By which α, γ and C may be accurately computed.

Examples.

1. Given $\beta = 15^\circ$, $B = 15^\circ\ 00'\ 10''$, solve the trihedral.

Ans $\alpha = 88^\circ\ 54'\ 37''$, $\gamma = 88^\circ\ 52'\ 18''$, $C = 89^\circ\ 42'\ 29''$;
$\alpha' = 91^\circ\ 5'\ 23''$, $\gamma' = 91^\circ\ 7'\ 42''$, $C' = 90^\circ\ 17'\ 31''$.

2. Given $\beta = 91^\circ$, $B = 90^\circ\ 59'$, solve the trihedral.

Ans. $\alpha = 89^\circ\ 49'\ 5''$, $\gamma = 100^\circ\ 28'\ 35''$, $C = 100^\circ\ 28'\ 29''$;
$\alpha' = 90^\circ\ 10'\ 55''$, $\gamma' = 79^\circ\ 31'\ 25''$, $C' = 79^\circ\ 31'\ 31''$.

124. Case VI. *Given the two oblique dihedrals; or B and C.*

The formulas, by the general method, are

$$\cos\alpha = \cot B \cot C, \ (a) \qquad \cos\beta = \frac{\cos B}{\sin C}, \ (b) \qquad \cos\gamma = \frac{\cos C}{\sin B}. \ (c)$$

Check, $\cos\alpha = \cos\beta\cos\gamma.$

If $B = 90°$, we have $\alpha = 90°$, $\beta = 90°$, and $\gamma = C$.

Examples.

1. Given $B = 47°\ 13'\ 43''$, $C = 126°\ 40'\ 24''$, solve the trihedral. *Ans.* $\alpha = 133°\,32'\,28''$, $\beta = 32°\,08'\,52''$, $\gamma = 144°\,27'$.

2. Given $B = 132°\ 46'\ 17''$, $C = 126°\ 40'\ 24''$, solve the trihedral. *Ans.* $\alpha = 46°\,27'\,32''$, $\beta = 147°\,51'\,08''$, $\gamma = 144°\,27'$.

3. Given $B = 132°\ 46'\ 17''$, $C = 53°\ 19'\ 36''$, solve the trihedral. *Ans.* $\alpha = 133°\,32'\,28''$, $\beta = 147°\,51'\,08''$, $\gamma = 35°\,33'$.

4. Given $B = 60°\ 47'\ 24''$, $C = 57°\ 16'\ 20''$, solve the trihedral. *Ans.* $\alpha = 68°\,56'\,29''$, $\beta = 54°\,32'\,32''$, $\gamma = 51°\,43'\,36''$.

125. If the above formulas do not give accurate results, the computations should be made with the following:

Reduce (a) by (5), and write formulas (a), (b) and (c),

$$\cos\alpha = \frac{\cos B\cos C}{\sin B\sin C}, \ (a) \qquad \cos\beta = \frac{\cos B}{\sin C}, \ (b) \qquad \cos\gamma = \frac{\cos C}{\sin B}. \ (c)$$

Subtract both members of each from unity, add both members to unity, divide and reduce, giving

$$\frac{1-\cos\alpha}{1+\cos\alpha} = \frac{\sin B\sin C - \cos B\cos C}{\sin B\sin C + \cos B\cos C}, \quad \ldots\ldots \quad (d)$$

$$\frac{1-\cos\beta}{1+\cos\beta} = \frac{\sin C - \cos B}{\sin C + \cos B} = \frac{\cos(90° - C) - \cos B}{\cos(90° - C) + \cos B}, \quad . \quad (e)$$

$$\frac{1-\cos\gamma}{1+\cos\gamma} = \frac{\sin B - \cos C}{\sin B + \cos C} = \frac{\cos(90° - B) - \cos C}{\cos(90° - B) + \cos C} \quad . \quad (f)$$

These, in (48), reducing (d) by (30) and (31), also (e) and (f) by (69), give

$$\tan \tfrac{1}{2}\alpha = \sqrt{\frac{-\cos(B+C)}{\cos(B-C)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

$$\tan \tfrac{1}{2}\beta = \sqrt{\tan \tfrac{1}{2}(B+C-90^\circ) \tan \tfrac{1}{2}(90^\circ + B - C)}. \quad (133)$$

$$\tan \tfrac{1}{2}\gamma = \sqrt{\tan \tfrac{1}{2}(B+C-90^\circ) \tan \tfrac{1}{2}(90^\circ - B + C)}. \quad (134)$$

SOLUTION OF RIGHT SPHERICAL TRIANGLES.

126. The radius of the sphere, and the given parts of the right spherical triangle, will determine the corresponding parts of its right trihedral. The computed parts of the trihedral, with the radius of the sphere, will determine the required parts of the triangle (§ 110).

There should be given, the radius of the sphere and two parts of the right spherical triangle, in addition to the right angle.

Examples.

In these examples let $A = 90^\circ$.

1. Given $R = 10$ feet, $a = 7$ feet, $b = 3$ feet, solve the triangle.

$$\alpha = \tfrac{7}{10} \text{ radians} = 40^\circ\ 06'\ 25'',$$
$$\beta = \tfrac{3}{10} \text{ radians} = 17^\circ\ 11'\ 19''.$$

Solving the trihedral, we have

$$\gamma = 36^\circ\ 48'\ 47'' = 0.64251 \text{ radians},$$
$$B = 27^\circ\ 18'\ 18'', \quad C = 68^\circ\ 26'\ 53'',$$
$$c = R\gamma = 6.425 \text{ ft}.$$

Hence the required parts of the triangle are

$$c = 6.425 \text{ ft.}, \quad B = 27^\circ\ 18'\ 18'', \quad C = 68^\circ\ 26'\ 53''.$$

2. Given $R = 100$, $b = 27.9253$, $c = 202.4582$, solve the triangle

Ans. $\gamma = 116^\circ$, $\beta = 16^\circ$, $\alpha = 114^\circ\ 55'.20''$, $a = 200.577$,
$B = 17^\circ\ 41'\ 40''$, $C = 97^\circ\ 39'\ 24''$.

3. Given $R = 50$, $a = 122.173$, $b = 17.4533$, solve the triangle.

Ans. $\alpha = 140°$, $\beta = 20°$, $\gamma = 144° \; 36' \; 30''$, $c = 126.194$,
$B = 32° \; 08' \; 48''$, $C = 115° \; 42' \; 23''$.

4. Given $R = 25$, $a = 30.2063$, $B = 99° \; 49' \; 30''$, solve the triangle.

Ans. $\alpha = 69° \; 13' \; 40''$, $\beta = 112° \; 53' \; 12''$, $\gamma = 155° \; 46' \; 46''$,
$b = 49.2561$, $c = 67.9716$, $C = 153° \; 58' \; 24''$.

5. Given $R = 100$, $b = 201.2946$, $B = 91° \; 1' \; 47''$, solve the triangle.

Ans. $\alpha = 64° \; 41' \; 10''$, $\beta = 115° \; 20'$, $\gamma = 177° \; 49' \; 27''$,
$\alpha' = 115° \; 18' \; 50''$, $\gamma' = 2° \; 10' \; 33''$,
$a = 112.8985$, $c = 310.3617$, $C = 177° \; 35' \; 35''$,
$a' = 201.2607$, $c' = 3.7975$, $C' = 2° \; 24' \; 25''$.

6. Given $R = 100$, $b = 201.2246$, $B = 88° \; 58' \; 13''$, solve the triangle. *Ans.* Impossible.

7. Given $R = 200$, $B = 47° \; 13' \; 43''$, $C = 126° \; 40' \; 24''$, solve the triangle.

Ans. $\alpha = 133° \; 32' \; 28''$, $\beta = 32° \; 08' \; 52''$, $\gamma = 144° \; 27'$,
$a = 466.1464$, $b = 112.2172$, $c = 504.2256$.

8. Given $R = 200$, $B = 37° \; 13' \; 43''$, $C = 127° \; 40' \; 24''$, solve the triangle. *Ans.* Impossible.

9. Given $R = 200$, $B = 157° \; 13' \; 43''$, $C = 126° \; 40' \; 24''$, solve the triangle. *Ans.* Impossible.

10. Given $R = 100$, $a = 69.8132$, $b = 104.7198$, solve the triangle. *Ans.* Impossible.

Isosceles Spherical Triangles and Trihedrals.

127. Any *isosceles spherical* triangle may be solved by drawing the arc of a great circle from the vertex to the middle point of the base. This arc is perpendicular to the base, and divides the triangle into two equal right spherical triangles.

Two unequal parts of the triangle will be sufficient to determine the triangle.

This method may also be used to solve any trihedral having two equal face-angles or two equal dihedrals.

Examples.

1. Given $R = 100$, $a = 225.7292$, $A = 177° \ 50' \ 26''$, solve the isosceles spherical triangle in which $b = c$.

Ans. $\alpha = 129° \ 20'$, $\beta = \gamma = 64° \ 41' \ 10''$,
or $115° \ 18' \ 50''$,
$B = C = 177° \ 35' \ 35''$ $b = c = 112.8985$
or $2° \ 24' \ 25''$, or 201.2607.

2. Given $R = 200$, $A = 74° \ 27' \ 26''$, $B = 127° \ 40' \ 24''$, solve the spherical triangle in which $b = c$. *Ans.* Impossible.

3. Given $R = 200$, $A = 94° \ 27' \ 26''$, $C = 126° \ 40' \ 24''$, solve the spherical triangle in which $b = c$.

Ans. $\alpha = 64° \ 17' \ 44''$, $\beta = \gamma = 133° \ 32' \ 28''$,
$a = 224.4344$, $b = c = 466.146$,
$B = 126° \ 40' \ 24''$.

QUADRANTAL SPHERICAL TRIANGLES AND TRIHEDRALS.

123. A *quadrantal spherical triangle* has one side equal to a quadrant. The corresponding trihedral is called a *quadrantal trihedral*, and has one face-angle equal to 90°.

Let A', B', C', a', b', c', α', β', γ', represent the parts of any quadrantal spherical triangle and its trihedral. Let $a' = \frac{1}{2}\pi R$, which gives $\alpha' = 90°$.

Let A, B, C, a, b, c, α, β, γ, represent the parts of the spherical triangle polar to the one just considered, and of its trihedral From Geometry, we have

$$\left.\begin{aligned} A + \alpha' &= 180°, \\ B + \beta' &= 180°, \\ C + \gamma' &= 180°, \end{aligned}\right\} \qquad \left.\begin{aligned} \alpha + A' &= 180°, \\ \beta + B' &= 180°, \\ \gamma + C' &= 180°. \end{aligned}\right\}$$

Any two parts of the quadrantal trihedral, in addition to $\alpha' = 90°$, will determine two parts of its *polar right trihedral*, in addition to $A = 90°$.

The right trihedral having been solved, the required parts of the quadrantal trihedral may be determined from the above relations.

Examples.

1. Given $\alpha' = 90°$, $B' = 75° \ 42'$, $\gamma' = 18° \ 37'$, solve the quadrantal trihedral.

Ans. $\alpha = 76° \ 25' \ 11''$, $\gamma = 161° \ 55' \ 20''$, $B = 94° \ 31' \ 21''$,
$A' = 103° \ 34' \ 49''$, $C' = 18° \ 04' \ 40''$, $\beta' = 85° \ 28' \ 39''$.

2. Given $\alpha' = 90°$, $B' = 24° \ 32' \ 06''$, $C' = 150° \ 13' \ 52''$, solve the trihedral.

Ans. $A' = 37° \ 50' \ 47''$, $\beta' = 42° \ 35' \ 39''$, $\gamma' = 125° \ 58' \ 44''$.

The quadrantal spherical triangle may be solved from its trihedral, if the radius of the sphere is given. Any two parts of the triangle, in addition to the quadrant, being given, will determine two parts of the trihedral in addition to $\alpha' = 90°$. The trihedral having been solved as above, its computed parts will determine the required parts of the triangle.

Example.

Given $b' = 1000$ mi., $c' = 500$ mi., $R = 400$ mi., solve the triangle. *Ans.* $\alpha' = 628.32$ mi. $A' = 63° \ 35' \ 23''$,
$B' = 147° \ 35' \ 22''$,
$C' = 58° \ 12' \ 24''$.

Applications.

1. A military covered-way has an interior slope of 45° inclination to the horizontal. A path, with grade of 5°, leads obliquely up the slope. What angle, along the surface of the slope, does the path make with the horizontal lower edge of the slope? *Ans.* 7° 5′.

2. In a pyramid with a square base, each edge makes an angle of 20° with the plane of the base. Find each lateral face-angle at the base, and the inclination of each plane face to the plane of the base. *Ans.* 48° 21′ 28″, 27° 14′ 10″.

3. In a pentagonal pyramid each lateral face-angle at the vertex is equal to 65°. Find the inclination of each lateral edge, and of each lateral plane face, to the plane of the base.
Ans. 23° 55′ 16″,
28° 44′ 09″.

4. Compute one dihedral angle in any regular tetrahedron.
Ans. 70° 31′ 43″.

5. In a hexagonal pyramid each plane face makes an angle of 75° with the plane of the base. Find the inclination of each edge to the plane of the base, and the value of each lateral face-angle at the base. *Ans.* 72° 48′ 29″. 81° 30′ 4″.

6. Given the radius of the earth 3963 miles, find the difference in longitude of two places whose common latitude is 52° 31′ 13″, the distance from one to the other, on the arc of a great circle, being 1000 miles. *Ans.* 23° 52′ 12″.

7. Find the length in geographical* miles, of the arc of a great circle joining Halifax—long. 63° 35′ w., lat. 44° 40′—with Cape Ferret (near Bordeaux, France), long. 1° 14′ w., lat. same as that of Halifax. *Ans.* 2592 miles.

8. Same as Example 7, except that distance is required in statute miles. *Ans.* 2988.3 mi.

9. Find the common latitude of two places on the earth's surface whose difference of longitude is 13° 10′, the arc of a great circle joining them being 693 miles. *Ans.* 40° 23′ 14″ N. or S.

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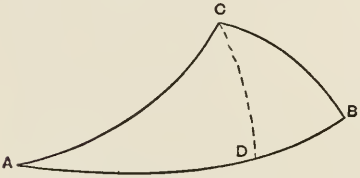
* A geographical mile is the length of an arc of 1′ on the earth's equator, or 1.15 + statute miles.

CHAPTER II.

OBLIQUE TRIHEDRALS AND SPHERICAL TRIANGLES.

129. *Solution by Right Spherical Triangles and Right Trihedrals.* If from a vertex C, of any oblique spherical triangle ABC, an arc of a great circle be drawn perpendicular to the opposite side AB, meeting it at D. two right spherical triangles will be formed. If the given parts of ABC determine two parts in one, and one part in the other, of these triangles they may be solved in succession, using the part common to the two right triangles as the second known part for the solution of the second triangle. Any oblique spherical triangle may be solved in this way when any three of its parts are given, except a, b and c, or A, B and C.



In general, the direct method is preferable, as it avoids the necessity of computing the perpendicular arc and its corresponding face-angle.

Relations of Parts in Oblique Trihedrals.

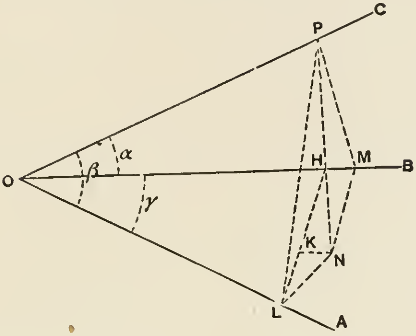
130. Let OA, OB, OC be the edges, and O the vertex of any oblique trihedral. Through P, any point of OC, pass two planes, one perpendicular to OA, intersecting AOC in PL, and AOB in LN; the other perpendicular to OB, intersecting BOC in MP, AOB in MN, and the first plane in PN, which is perpendicular to OAB and hence to LN and MN.

In the planes PLN and AOC, we have

$$NP = LP \sin A = OP \sin \beta \sin A.$$

In the planes PMN and BOC, we have

$$NP = MP \sin B = OP \sin \alpha \sin B.$$



Equating these values of NP, we have

$$\sin \alpha \sin B = \sin \beta \sin A;$$

or,
$$\frac{\sin \alpha}{\sin A} = \frac{\sin \beta}{\sin B}. \qquad \text{(135)}$$

Similarly,
$$\frac{\sin \alpha}{\sin A} = \frac{\sin \gamma}{\sin C}. \qquad \text{(136)}$$

That is, *the sines of the face-angles are proportional to the sines of their opposite dihedrals.*

131. Through L draw LH perpendicular to OB. Through N draw NK perpendicular to LH.

$$\cos \alpha = \frac{OM}{OP} = \frac{OH}{OP} + \frac{HM}{OP}.$$

$$OH = OL \cos \gamma = OP \cos \beta \cos \gamma.$$

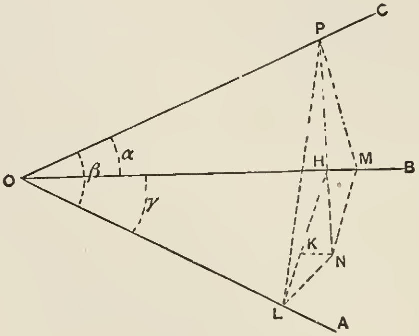
$NMHK$ is a parallelogram. Hence,

$$HM = KN.$$

The triangles LKN and OHL have their homologous sides perpendicular, each to each, and are therefore similar.

Hence, $$\frac{KN}{LN} = \frac{HL}{OL} = \sin \gamma.$$

and $$KN = LN \sin \gamma = LP \cos A \sin \gamma = OP \sin \beta \cos A \sin \gamma.$$



Therefore,

$$HM = OP \sin \beta \sin \gamma \cos A.$$

Hence, $$\cos \alpha = \cos \beta \cos \gamma + \sin \beta \sin \gamma \cos \text{A}. \quad . \quad . \quad . \qquad (137)$$

Similarly, $$\cos \beta = \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \text{B}, \quad . \quad . \quad . \qquad (138)$$

$$\cos \gamma = \cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \text{C}. \quad . \quad . \quad . \qquad (139)$$

That is: *the cosine of any face-angle is equal to the product of the cosines of the other two face-angles, plus the product of their sines multiplied by the cosine of their included dihedral.*

132. Resume the preceding notation, and denote by A', B', C', α', β', γ', respectively, the dihedrals and face-angles of the corresponding polar trihedral.

From (137), (138) and (139), we have

$$\left.\begin{aligned} \cos \alpha' &= \cos \beta' \cos \gamma' + \sin \beta' \sin \gamma' \cos A', \\ \cos \beta' &= \cos \alpha' \cos \gamma' + \sin \alpha' \sin \gamma' \cos B', \\ \cos \gamma' &= \cos \alpha' \cos \beta' + \sin \alpha' \sin \beta' \cos C'. \end{aligned}\right\} \ldots \quad (a)$$

But by Geometry,

$$\alpha' = 180^\circ - A, \quad \beta' = 180^\circ - B, \quad \gamma' = 180^\circ - C,$$
$$A' = 180^\circ - \alpha, \quad B' = 180^\circ - \beta, \quad C' = 180^\circ - \gamma;$$

$$\begin{aligned} \therefore \sin \alpha' &= \sin A, & \cos \alpha' &= -\cos A, \\ \sin \beta' &= \sin B, & \cos \beta' &= -\cos B, \\ \text{etc.} &= \text{etc.} & \text{etc.} &= \text{etc.} \end{aligned}$$

Which in (a), after multiplying both members by -1, give

$$\cos \mathbf{A} = -\cos \mathbf{B} \cos \mathbf{C} + \sin \mathbf{B} \sin \mathbf{C} \cos \alpha, \quad . \quad . \quad (140)$$

$$\cos \mathbf{B} = -\cos \mathbf{A} \cos \mathbf{C} + \sin \mathbf{A} \sin \mathbf{C} \cos \beta, \quad . \quad . \quad (141)$$

$$\cos \mathbf{C} = -\cos \mathbf{A} \cos \mathbf{B} + \sin \mathbf{A} \sin \mathbf{B} \cos \gamma. \quad . \quad . \quad (142)$$

That is: *the cosine of any dihedral is equal to minus the product of the cosines of the other two dihedrals, plus the product of their sines multiplied by the cosine of their included face-angle.*

133. The construction being the same as in § 131, we have

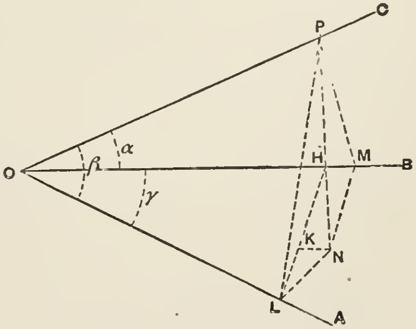
$$LH = LK + NM. \quad (a)$$

The right triangle OHL gives

$$\begin{aligned} LH &= LO \sin \gamma \\ &= OP \cos \beta \sin \gamma. \end{aligned}$$

The right triangle NKL gives

$$\begin{aligned} LK &= NL \cos \gamma \\ &= PL \cos A \cos \gamma \\ &= OP \sin \beta \cos \gamma \cos A. \end{aligned}$$



The right triangle PNM gives

$$NM = NP \cot B = PL \sin A \cot B = OP \sin \beta \sin A \cot B.$$

Hence, substituting in (a), and dividing both members by $OP \sin \beta$, we have

$$\cot \beta \sin \gamma = \cos \gamma \cos A + \sin A \cot B.$$

Similarly, we obtain

$$\cot \alpha \sin \gamma = \cos \gamma \cos B + \sin B \cot A,$$
$$\cot \gamma \sin \beta = \cos \beta \cos A + \sin A \cot C.$$

134. Solving (137) with respect to cos A, we have

$$\cos A = \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

From which A may be computed, knowing α, β and γ.

To deduce a more convenient formula for logarithmic computation, subtract both members from unity, add both to unity, substitute in (44) and (46) respectively, and we have

$$2 \sin^2 \tfrac{1}{2}A = 1 - \cos A = \frac{\sin \beta \sin \gamma - \cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma},$$

$$2 \cos^2 \tfrac{1}{2}A = 1 + \cos A = \frac{\sin \beta \sin \gamma + \cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}.$$

Reducing by (31) and (30), and then by (63), we have

$$2\sin^2\tfrac{1}{2}A = \frac{\cos(\beta-\gamma)-\cos\alpha}{\sin\beta\sin\gamma} = \frac{2\sin\tfrac{1}{2}(\alpha+\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\beta+\gamma)}{\sin\beta\sin\gamma}, \quad (a)$$

$$2\cos^2\tfrac{1}{2}A = \frac{\cos\alpha-\cos(\beta+\gamma)}{\sin\beta\sin\gamma} = \frac{2\sin\tfrac{1}{2}(\beta+\gamma+\alpha)\sin\tfrac{1}{2}(\beta+\gamma-\alpha)}{\sin\beta\sin\gamma}. \quad (b)$$

Let $$\alpha+\beta+\gamma = s,$$

then $$\beta+\gamma-\alpha = s-2\alpha, \quad \alpha-\beta+\gamma = s-2\beta,$$

and $$\alpha+\beta-\gamma = s-2\gamma;$$

which in (*a*) and (*b*), after dividing both members by 2 and extracting the square roots of both, give

$$\sin\tfrac{1}{2}A = \sqrt{\frac{\sin(\tfrac{1}{2}s-\beta)\sin(\tfrac{1}{2}s-\gamma)}{\sin\beta\sin\gamma}}, \quad . \quad . \quad . \quad (144)$$

$$\cos\tfrac{1}{2}A = \sqrt{\frac{\sin\tfrac{1}{2}s\sin(\tfrac{1}{2}s-\alpha)}{\sin\beta\sin\gamma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

Similarly, from (48), or by division from these, we have

$$\tan\tfrac{1}{2}A = \sqrt{\frac{\sin(\tfrac{1}{2}s-\beta)\sin(\tfrac{1}{2}s-\gamma)}{\sin\tfrac{1}{2}s\sin(\tfrac{1}{2}s-\alpha)}}, \quad . \quad . \quad . \quad (146)$$

which, in general, is preferable.

If more than one angle is required, let

$$h = \sqrt{\frac{\sin(\frac{1}{2}s - \alpha)\sin(\frac{1}{2}s - \beta)\sin(\frac{1}{2}s - \gamma)}{\sin\frac{1}{2}s}}, \quad . \quad (147)$$

which is symmetrical with respect to α, β and γ. It gives

$$\tan\tfrac{1}{2}A = \frac{h}{\sin(\frac{1}{2}s - \alpha)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

The plus sign is used in the second member of each of these formulas, since $\frac{1}{2}A < 90°$.

135. From (140) we have

$$\cos\alpha = \frac{\cos B\cos C + \cos A}{\sin B\sin C}. \quad . \quad . \quad . \quad . \quad . \quad (149)$$

from which α may be computed if A, B and C are given.

For convenience of logarithmic computation, (149) is transformed as was (143) in § 134; and assuming

$$A + B + C = S,$$

we have

$$\sin\tfrac{1}{2}\alpha = \sqrt{\frac{-\cos\frac{1}{2}S\cos(\frac{1}{2}S - A)}{\sin B\sin C}}, \quad . \quad . \quad . \quad . \quad (150)$$

$$\cos\tfrac{1}{2}\alpha = \sqrt{\frac{\cos(\frac{1}{2}S - B)\cdot\cos(\frac{1}{2}S - C)}{\sin B\sin C}}, \quad . \quad . \quad . \quad (151)$$

$$\tan\tfrac{1}{2}\alpha = \sqrt{\frac{-\cos\frac{1}{2}S\cos(\frac{1}{2}S - A)}{\cos(\frac{1}{2}S - B)\cos(\frac{1}{2}S - C)}}. \quad . \quad . \quad . \quad (152)$$

Using the sign + in each second member, since $\frac{1}{2}\alpha < 90°$. Placing

$$H = \sqrt{\frac{-\cos \frac{1}{2}S}{\cos(\frac{1}{2}S - A)\cos(\frac{1}{2}S - B)\cos(\frac{1}{2}S - C)}}, \quad . \qquad (153)$$

which is symmetrical with reference to A, B and C, we have

$$\tan \tfrac{1}{2}\alpha = H \cos(\tfrac{1}{2}S - A). \quad . \quad . \quad . \quad . \quad . \qquad (154)$$

Equations (150), (151) and (152) may also be formed by applying (144), (145) and (146) to the polar trihedral.

136. *Gauss' Equations.*

$$\sin \tfrac{1}{2}(A \pm B) = \sin(\tfrac{1}{2}A \pm \tfrac{1}{2}B),$$

or, by (32),

$$= \sin \tfrac{1}{2}A \cos \tfrac{1}{2}B \pm \cos \tfrac{1}{2}A \sin \tfrac{1}{2}B,$$

and, by (144) and (145),

$$= \sqrt{\frac{\sin(\frac{1}{2}s - \beta)\sin(\frac{1}{2}s - \gamma)}{\sin\beta \sin\gamma} \cdot \frac{\sin\frac{1}{2}s \sin(\frac{1}{2}s - \beta)}{\sin\alpha \sin\gamma}}$$

$$\pm \sqrt{\frac{\sin\frac{1}{2}s \sin(\frac{1}{2}s - \alpha)}{\sin\beta \sin\gamma} \cdot \frac{\sin(\frac{1}{2}s - \alpha)\sin(\frac{1}{2}s - \gamma)}{\sin\alpha \sin\gamma}}.$$

$$= \frac{\sin(\frac{1}{2}s - \beta)}{\sin\gamma}\sqrt{\frac{\sin\frac{1}{2}s \sin(\frac{1}{2}s - \gamma)}{\sin\alpha \sin\beta}}$$

$$\pm \frac{\sin(\frac{1}{2}s - \alpha)}{\sin\gamma}\sqrt{\frac{\sin\frac{1}{2}s \sin(\frac{1}{2}s - \gamma)}{\sin\alpha \sin\beta}};$$

or reducing by (145), we have

$$\sin \tfrac{1}{2}(A \pm B) = \frac{\sin (\tfrac{1}{2}s - \beta) \pm \sin (\tfrac{1}{2}s - \alpha)}{\sin \gamma} \cos \tfrac{1}{2}C.$$

But by (60), (61) and (36), we have

$$\sin (\tfrac{1}{2}s - \beta) + \sin (\tfrac{1}{2}s - \alpha) = 2 \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}(\alpha - \beta),$$

$$\sin (\tfrac{1}{2}s - \beta) - \sin (\tfrac{1}{2}s - \alpha) = 2 \cos \tfrac{1}{2}\gamma \sin \tfrac{1}{2}(\alpha - \beta),$$

and $$\sin \gamma = 2 \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\gamma.$$

Hence, $$\sin \tfrac{1}{2}(A + B) = \frac{\cos \tfrac{1}{2}(\alpha - \beta)}{\cos \tfrac{1}{2}\gamma} \cos \tfrac{1}{2}C, \qquad (155)$$

$$\sin \tfrac{1}{2}(A - B) = \frac{\sin \tfrac{1}{2}(\alpha - \beta)}{\sin \tfrac{1}{2}\gamma} \cos \tfrac{1}{2}C. \qquad (156)$$

Similarly,

$$\cos \tfrac{1}{2}(A \pm B) = \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \mp \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B$$

$$= \sqrt{\frac{\sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - \alpha)}{\sin \beta \sin \gamma} \cdot \frac{\sin \tfrac{1}{2}s \sin (\tfrac{1}{2}s - \beta)}{\sin \alpha \sin \gamma}}$$

$$\mp \sqrt{\frac{\sin (\tfrac{1}{2}s - \beta) \sin (\tfrac{1}{2}s - \gamma)}{\sin \beta \sin \gamma} \cdot \frac{\sin (\tfrac{1}{2}s - \alpha) \sin (\tfrac{1}{2}s - \gamma)}{\sin \alpha \sin \gamma}}$$

$$= \left(\frac{\sin \tfrac{1}{2}s}{\sin \gamma} \mp \frac{\sin (\tfrac{1}{2}s - \gamma)}{\sin \gamma}\right) \sqrt{\frac{\sin (\tfrac{1}{2}s - \alpha) \sin (\tfrac{1}{2}s - \beta)}{\sin \alpha \sin \beta}}$$

$$= \frac{\sin \frac{1}{2}s \mp \sin (\frac{1}{2}s - \gamma)}{\sin \gamma} \sin \tfrac{1}{2}C,$$

giving $$\cos \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}\gamma} \sin \tfrac{1}{2}C. \quad . \quad . \quad . \qquad (157)$$

$$\cos \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}\gamma} \sin \tfrac{1}{2}C. \quad . \quad . \quad . \qquad (158)$$

Equations (155), (156), (157) and (158) are known as *Gauss' Equations.*

137. *Napier's Analogies:* From (155) and (157), by division, we have

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)} \cot \tfrac{1}{2}C,$$

or $$\cos \tfrac{1}{2}(\alpha + \beta) : \cos \tfrac{1}{2}(\alpha - \beta) :: \cot \tfrac{1}{2}\mathbf{C} : \tan \tfrac{1}{2}(\mathbf{A} + \mathbf{B}). \quad . \qquad (159)$$

Similarly, from (156) and (158),

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)} \cot \tfrac{1}{2}C,$$

or $$\sin \tfrac{1}{2}(\alpha + \beta) : \sin \tfrac{1}{2}(\alpha - \beta) :: \cot \tfrac{1}{2}\mathbf{C} : \tan \tfrac{1}{2}(\mathbf{A} - \mathbf{B}). \quad . \qquad (160)$$

From (157) and (158),

$$\frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} = \frac{\tan \frac{1}{2}\gamma}{\tan \frac{1}{2}(\alpha + \beta)},$$

or $$\cos \tfrac{1}{2}(\mathbf{A} + \mathbf{B}) : \cos \tfrac{1}{2}(\mathbf{A} - \mathbf{B}) :: \tan \tfrac{1}{2}\gamma : \tan \tfrac{1}{2}(\alpha + \beta); \quad . \qquad (161)$$

and from (155) and (156),

$$\frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} = \frac{\tan \frac{1}{2}\gamma}{\tan \frac{1}{2}(\alpha - \beta)},$$

or $$\sin \tfrac{1}{2}(\mathrm{A} + \mathrm{B}) : \sin \tfrac{1}{2}(\mathrm{A} - \mathrm{B}) :: \tan \tfrac{1}{2}\gamma : \tan \tfrac{1}{2}(\alpha - \beta). \quad . \quad (162)$$

These four proportions are known as *Napier's Analogies.*

PRINCIPLES OF OBLIQUE TRIHEDRALS.

138. In every trihedral considered:

I. *Each part is less than* 180° (§111).

II. *The sum of the face-angles is less than* 360°.

III. *Each face-angle is less than the sum, and greater than the difference, of the other two face-angles.*

IV. *The sum of the dihedrals is greater than* 180°, *and less than* 540°.

V. *Each dihedral is greater than the difference between* 180° *and the sum of the other two, and less than the difference between* 180° *and the difference of the other two.*

By the 4th principle we have

$$A + B + C > 180^\circ.$$

Hence, $$A > 180^\circ - (B + C). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The 3d principle applied to the polar trihedral gives

$$\alpha' < \beta' + \gamma',$$

or, passing to the trihedral considered,

$$180° - A < 180° - B + 180° - C,$$

or $$A > B + C - 180°. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Similarly,

$$C > A + B - 180°, \qquad \text{and } B > A + C - 180°,$$

from which, by transposition, we have

$$A < 180° - (B - C) \qquad \text{and} \qquad A < 180° - (C - B),$$

which, with relations (a) and (b), establish the principle.

VI. *A greater face-angle lies opposite to a greater dihedral, and vice versa.*

For, by Geometry, a greater side lies opposite to a greater angle in the corresponding spherical triangle.

VII. *An oblique face-angle which differs from* 90° *as much as, or more than, another face-angle, is of the same species as its opposite dihedral.* From (137),

$$\cos \alpha - \cos \beta \cos \gamma = \sin \beta \sin \gamma \cos A. \quad . \quad . \quad . \quad (c)$$

Let α differ from 90° as much as, or more than β. Then $\cos \alpha$ is numerically equal to, or numerically greater than $\cos \beta$; and since $\cos \gamma$ cannot exceed unity, $\cos \alpha$ is numerically equal to, or greater than $\cos \beta \cos \gamma$.

The first member of (c) can be zero only when $A = 90°$ and $\gamma = 90°$, which requires $\alpha = 90°$ (§ 117).

Hence, if α is oblique, $\cos \alpha$ determines the algebraic sign of the first member, and must have the same sign as $\cos A$, which determines the sign of the second member. Hence, α and A are of the same species.

VIII. *An oblique dihedral which differs from 90° as much as, or more than, another dihedral, is of the same species as its opposite face-angle.* From (140) we have

$$\cos A + \cos B \cos C = \sin B \sin C \cos \alpha,$$

from which a proof entirely analogous to that of the preceding principle may be made.

IX. *At least two face-angles must be of the same species as their opposite dihedrals.*

This follows from either the 7th or the 8th principle, unless the trihedral is bi- or tri-rectangular.

X. *The half-sum of any two face-angles is of the same species as the half-sum of their opposite dihedrals.*

(159) gives

$$\cos \tfrac{1}{2}(\alpha + \beta) \tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2}C \cos \tfrac{1}{2}(\alpha - \beta),$$

in which, by the 1st principle, $\frac{1}{2}C$ and $\frac{1}{2}(\alpha - \beta)$ are each $< 90°$. Hence, the second member is positive, and $\cos \frac{1}{2}(\alpha + \beta)$ has the same algebraic sign as $\tan \frac{1}{2}(A + B)$. This requires $\frac{1}{2}(\alpha + \beta)$ to be of the same species as $\frac{1}{2}(A + B)$.

If $\frac{1}{2}(\alpha + \beta) = 90°$, we may write, (159),

$$\tan \tfrac{1}{2}(A + B) = \frac{\cot \frac{1}{2}C \cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)},$$

which gives $\tan \frac{1}{2}(A + B) = \infty$, or $\frac{1}{2}(A + B) = 90°$.

If $\frac{1}{2}(A + B) = 90°$, it may be shown that $\frac{1}{2}(\alpha + \beta) = 90°$.

139. *Solution of Oblique Trihedrals* There are but three independent equations among the formulas which express relations between the six parts of any trihedral. Any three parts will, therefore, determine the others.

There may be given—

I. Three face-angles.
II. Two face-angles and their included dihedral.
III. Two face-angles and a dihedral opposite to one of them.
IV. Two dihedrals and a face-angle opposite to one of them.
V. Two dihedrals and their included face-angle.
VI. Three dihedrals.

In every case the three *given parts should be tested* by the principles of § 138, which should be satisfied before attempting any solution.

Unless otherwise directed, the required parts should be computed directly from the given parts by means of the foregoing formulas.

140. CASE I. *Given the three face-angles;* or α, β and γ.

Either of formulas (144), (145) or (146) may be used. The last is usually preferred; and if more than one dihedral is required, its modification, expressed in (148), as follows,

$$\tan \tfrac{1}{2}A = \frac{h}{\sin(\frac{1}{2}s - \alpha)},$$

in which, $h = \sqrt{\dfrac{\sin(\frac{1}{2}s - \alpha)\sin(\frac{1}{2}s - \beta)\sin(\frac{1}{2}s - \gamma)}{\sin\frac{1}{2}s}}$,

should be used.

By principle II, $(\frac{1}{2}s - \alpha) < 180°$, and it is positive by principle III. Hence, each factor under the radical sign is always positive, and h is *always real.*

Examples.

1. Given $\alpha = 101° 09' 10''$, $\beta = 108° 43' 30''$, $\gamma = 150° 05' 20''$, solve the trihedral.

Solution.

$\alpha = 101° 09' 10''$ $\qquad \alpha < \beta + \gamma, \quad \alpha > \gamma - \beta.$
$\beta = 108° 43' 30''$
$\gamma = 150° 05' 20''$ $\qquad h = \sqrt{\dfrac{\sin(\frac{1}{2}s - \alpha)\sin(\frac{1}{2}s - \beta)\sin(\frac{1}{2}s - \gamma)}{\sin\frac{1}{2}s}}$
$s \quad 359° 58' 00''$

$$\begin{aligned} \tfrac{1}{2}s &= 179^\circ\ 59'\ 00'' & \text{a. c. }\log\sin\tfrac{1}{2}s &= 3.5362\hat{7} \\ \tfrac{1}{2}s-\alpha &= 78^\circ\ 49'\ 50'' & \log\sin(\tfrac{1}{2}s-\alpha) &= 9.9916\hat{9} \\ \tfrac{1}{2}s-\beta &= 71^\circ\ 15'\ 30'' & \log\sin(\tfrac{1}{2}s-\beta) &= 9.97634 \\ \tfrac{1}{2}s-\gamma &= 29^\circ\ 53'\ 40'' & \log\sin(\tfrac{1}{2}s-\gamma) &= \underline{9.69758} \\ & & & \quad 3.20189)2 \\ & & \log h &= 1.6009\hat{4} \end{aligned}$$

$$\tan\tfrac{1}{2}A = \frac{h}{\sin(\tfrac{1}{2}s-\alpha)},\quad \tan\tfrac{1}{2}B = \frac{h}{\sin(\tfrac{1}{2}s-\beta)},\quad \tan C = \frac{h}{\sin(\tfrac{1}{2}s-\gamma)}.$$

$$\begin{aligned} \log h &= 1.6009\hat{4} & \log h &= 1.6009\hat{4} \\ \text{a. c. }\log\sin(\tfrac{1}{2}s-\alpha) &= 0.0083\hat{6} & \text{a. c. }\log\sin(\tfrac{1}{2}s-\beta) &= 0.02366 \\ \log\tan\tfrac{1}{2}A &= 1.60925 & \log\tan\tfrac{1}{2}B &= 1.6246\hat{6} \\ & 88^\circ\ 35'\ 29'' & & 88^\circ\ 38'\ 25'' \\ \log h &= 1.6009\hat{4} & & \\ \text{a. c. }\log\sin(\tfrac{1}{2}s-\gamma) &= 0.30242 & A &= 177^\circ\ 10'\ 58'', \\ & & B &= 177^\circ\ 16'\ 50'', \\ \log\tan\tfrac{1}{2}C &= 1.9033\hat{6} & C &= 178^\circ\ 34'\ 06''. \\ & 89^\circ\ 17'\ 03'' & & \end{aligned}$$

2. Given $\alpha = 74^\circ\ 24'$, $\beta = 35^\circ\ 46'\ 14''$, $\gamma = 100^\circ\ 39'$, solve the trihedral.

Ans. $A = 43^\circ\ 11'\ 10''$, $B = 24^\circ\ 32'\ 28''$, $C = 135^\circ\ 42'\ 32''$.

3. Given $\alpha = 76^\circ\ 29'\ 18''$, $\beta = 93^\circ\ 18'\ 36''$, $\gamma = 122^\circ\ 7'\ 42''$; find $C = 122^\circ\ 16'\ 24''$.

141. Case II. *Given two face-angles and their included dihedral;* or α, β and C.

To compute A and B, we have the first and second of Napier's Analogies (159) and (160), as follows:

$$\cos\tfrac{1}{2}(\alpha+\beta) : \cos\tfrac{1}{2}(\alpha-\beta) :: \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A+B),$$
$$\sin\tfrac{1}{2}(\alpha+\beta) : \sin\tfrac{1}{2}(\alpha-\beta) :: \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A-B).$$

The former will determine $\frac{1}{2}(A+B)$, the latter $\frac{1}{2}(A-B)$. Then we have

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B),$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

Care should be taken to ascertain the algebraic signs of $\tan \frac{1}{2}(A+B)$ and $\tan \frac{1}{2}(A-B)$, in order to determine the species of $\frac{1}{2}(A+B)$ and the sign of $\frac{1}{2}(A-B)$. If the greater value be always denoted by α, and the lesser by β, it will be sufficient to find the species of $\frac{1}{2}(A+B)$ from § 138, X.

To compute γ, we may use the third, (161), or fourth, (162), of Napier's Analogies, or the proportion of the sines (136). The last should not be used if C is nearer to 90° than both A and B, as the 8th principle of § 138 would then fail to determine the species of γ. It may always be used as a check.

Examples.

1. Given $\alpha = 62° \, 38'$, $\beta = 10° \, 13' \, 20''$, $C = 150° \, 24' \, 12''$, solve the trihedral.

$$\tan \tfrac{1}{2}(A+B) = \frac{\cos \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)} \cot \tfrac{1}{2}C, \quad \tan \tfrac{1}{2}(A-B) = \frac{\sin \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}(\alpha+\beta)} \cot \tfrac{1}{2}C.$$

$$\begin{aligned} \tfrac{1}{2}(\alpha+\beta) &= 36° \, 25' \, 40'', \\ \tfrac{1}{2}(\alpha-\beta) &= 26° \, 12' \, 20'', \\ \tfrac{1}{2}C &= 75° \, 12' \, 06'' \end{aligned}$$

$$\begin{array}{rlrl} \text{a. c. } \log \cos \frac{1}{2}(\alpha+\beta) &= 0.09442 & \text{a. c. } \log \sin \frac{1}{2}(\alpha+\beta) &= 0.22635 \\ + \log \cos \frac{1}{2}(\alpha-\beta) &= 9.95290 & + \log \sin \frac{1}{2}(\alpha-\beta) &= 9.64502 \\ + \log \cot \frac{1}{2}C &= \underline{9.42190} & + \log \cot \frac{1}{2}C &= \underline{9.42190} \\ \log \tan \frac{1}{2}(A+B) &= 9.46922 & \log \tan \frac{1}{2}(A-B) &= 9.29327 \\ & 16° \, 24' \, 52'' & & 11° \, 06' \, 53'' \\ \frac{1}{2}(A+B) &= 16° \, 24' \, 52'' & \frac{1}{2}(A-B) &= 11° \, 06' \, 53'' \\ A &= 27° \, 31' \, 45'' & B &= 5° \, 17' \, 59'' \end{array}$$

To compute γ, we have, by (161),

$$\tan \tfrac{1}{2}\gamma = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(\alpha + \beta).$$

$$\begin{array}{rl}
\text{a. c. } \log \cos \frac{1}{2}(A - B) &= 0.00822 \\
+ \log \cos \frac{1}{2}(A + B) &= 9.98193 \\
+ \log \tan \frac{1}{2}(\alpha + \beta) &= \underline{9.86806} \\
\log \tan \frac{1}{2}\gamma \quad &= 9.85821 \\
& 35^\circ\ 48'\ 32'' \\
\gamma = & 71^\circ\ 37'\ 04''
\end{array}$$

Check, $$\frac{\sin \alpha}{\sin A} = \frac{\sin \beta}{\sin B} = \frac{\sin \gamma}{\sin C}$$

$$\begin{array}{lll}
\log \sin \alpha = 9.9484\hat{5} & \log \sin \beta = 9.24912 & \log \sin \gamma = 9.9772\hat{5} \\
\text{a.c. } \log \sin A = \underline{0.33517} & \text{a.c. } \log \sin B = \underline{1.03449} & \text{a.c. } \log \sin C = \underline{0.30637} \\
\qquad 0.2836\hat{2} & \qquad 0.28361 & \qquad 0.2836\hat{2}
\end{array}$$

This discrepancy corresponds to an error of only 0″.6 in B. More accurately $\frac{1}{2}(A + B) = 16^\circ\ 24'\ 51''.5$, $\frac{1}{2}(A - B) = 11^\circ\ 6'\ 53''.3$ and $B = 5^\circ\ 17'\ 58''.2$.

2. Given $\alpha = 68^\circ\ 46'\ 2''$, $\beta = 37^\circ\ 10'$, $C = 39^\circ\ 23'$, find $A = 120^\circ\ 59'\ 47''$, $B = 33^\circ\ 45'\ 3''$, $\gamma = 43^\circ\ 37'\ 38''$

3. Given $\alpha = 132\ 46'\ 42''$, $\beta = 59^\circ\ 50'\ 06''$, $C = 56^\circ\ 28'\ 24''$, find $A = 142^\circ\ 16'\ 23''$, $B = 46^\circ\ 7'\ 5''$.

142. CASE III. *Given two face-angles and a dihedral opposite one of them;* or α, β and A.

To compute B. (135) gives

$$\sin B = \frac{\sin \beta \sin A}{\sin \alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Hence, $\log \sin B = \log \sin \beta + \log \sin A - \log \sin \alpha.$

1°. If $\log \sin B > 0$; $\sin B > 1$, which is impossible, and there is *no solution.*

2°. If $\log \sin B = 0$; $\sin B = 1$, and $B = 90°$, giving but *one possible solution*, a right trihedral.

3°. If $\log \sin B < 0$*; $\sin B < 1$, giving two supplementary angles, which we denote by B and B''.

If β is not nearer to 90° than α, B must be of the same species as β, giving *one solution* only.

If β is nearer to 90° than α, both B and B'' are admissible, giving, as we shall see, *two solutions*.

To compute C and γ. We have, from (160) and (162),

$$\cot \tfrac{1}{2}C = \frac{\sin \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)} \tan \tfrac{1}{2}(A - B),$$

$$\tan \tfrac{1}{2}\gamma = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(\alpha - \beta).$$

To show that the above formulas give double admissible values for C and γ when we have double admissible values for B, we note that, in this case, α and A are of the same species because β is nearer 90° than α; also $\sin \alpha < \sin \beta$, giving in (a)

$$\sin A < \sin B \text{ or } \sin B''.$$

Therefore, A is farther from 90° than B or B''. Hence—

1st. If $\alpha > \beta$, we have

$$\alpha > 90°, A > 90°, \text{ and } A > B \text{ or } B''.$$

2d. If $\alpha < \beta$, then

$$\alpha < 90°, A < 90°, \text{ and } A < B \text{ or } B''.$$

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* Practically $\log \sin B$ cannot be less than -10.

In either case the above formulas give positive values for cot $\frac{1}{2}C$ and tan $\frac{1}{2}\gamma$ with B or B''. Hence, both values of C and γ are admissible, giving two solutions.

The values found for C and γ may be checked by (136).

Examples.

1. Given $\alpha = 79^\circ\ 12'\ 10''$, $\beta = 119^\circ\ 03'\ 26''$, $A = 58^\circ\ 08'\ 00''$, solve the trihedral.

Solution.

$$\sin B = \sin\beta\,\frac{\sin A}{\sin\alpha}$$

$$\begin{aligned} \log\sin\beta &= 9.94158 \\ \log\sin A &= 9.92905 \\ \text{a. c. } \log\sin\alpha &= 0.0077\hat{5} \\ \hline \log\sin B &= 9.8783\hat{8} \\ &\ 49^\circ\ 05'\ 30'' \end{aligned}$$

$$B = 130^\circ\ 54'\ 30''\ [\S\ 138,\ \text{VII. or VI.}].$$

$$\cot\tfrac{1}{2}C = \frac{\sin\frac{1}{2}(\beta+\alpha)}{\sin\frac{1}{2}(\beta-\alpha)}\tan\tfrac{1}{2}(B-A),$$

$$\tan\tfrac{1}{2}\gamma = \frac{\sin\frac{1}{2}(B+A)}{\sin\frac{1}{2}(B-A)}\tan\tfrac{1}{2}(\beta-\alpha).$$

$$\begin{aligned} \beta+\alpha &= 198^\circ\ 15'\ 36'' & \tfrac{1}{2}(\beta+\alpha) &= 99^\circ\ 07'\ 48'' \\ \beta-\alpha &= 39^\circ\ 51'\ 16'' & \tfrac{1}{2}(\beta-\alpha) &= 19^\circ\ 55'\ 38'' \\ B+A &= 189^\circ\ 02'\ 30'' & \tfrac{1}{2}(B+A) &= 94^\circ\ 31'\ 15'' \\ B-A &= 72^\circ\ 46'\ 30'' & \tfrac{1}{2}(B-A) &= 36^\circ\ 23'\ 15'' \end{aligned}$$

$$\begin{aligned} \log\tan\tfrac{1}{2}(B-A) &= 9.8674\hat{2} & \log\tan\tfrac{1}{2}(\beta-\alpha) &= 9.5593\hat{4} \\ \log\sin\tfrac{1}{2}(\beta+\alpha) &= 9.9944\hat{6} & \log\sin\tfrac{1}{2}(B+A) &= 9.99865 \\ \text{a. c. } \log\sin\tfrac{1}{2}(\beta-\alpha) &= 0.46747 & \text{a.c. } \log\sin\tfrac{1}{2}(B-A) &= 0.2267\hat{6} \\ \hline \log\cot\tfrac{1}{2}C &= 0.32936 & \log\tan\tfrac{1}{2}\gamma &= 9.78476 \\ \tfrac{1}{2}C &= 25^\circ\ 05'\ 58'' & \tfrac{1}{2}\gamma &= 31^\circ\ 20'\ 59'' \\ C &= 50^\circ\ 11'\ 56'' & \gamma &= 62^\circ\ 41'\ 58'' \end{aligned}$$

Check, $$\frac{\sin C}{\sin \gamma} = \frac{\sin A}{\sin \alpha}.$$

log sin C	$9.8855\hat{1}$	log sin A	9.92905
a. c. log sin γ	0.05129	a. c. log sin α	$0.0077\hat{5}$
	$9.9368\hat{6}$		$9.9368\hat{6}$

2. Given $\alpha = 43° \; 27' \; 36''$, $\beta = 82° \; 58' \; 17''$, $A = 29° \; 32' \; 29''$, solve the trihedral.

Ans. $B = 45° \; 21' \; 01''$, $C = 139° 43' \; 30''$, $\gamma = 115° \; 35' \; 48.''$;
$B'' = 134° \; 38' \; 59''$, $C' = 32° \; 20' \; 28''$, $\gamma' = 48° \; 16' \; 18''$.

3. Given $\alpha = 136° \; 32' \; 24''$, $\beta = 82° \; 58' \; 17''$, $A = 29° \; 32' \; 29''$, solve the trihedral. *Ans.* Impossible [§ 138, VII.].

4. Given $\alpha = 27° \; 31' \; 27''$, $\beta = 82° \; 58' \; 17''$, $A = 29° \; 32' \; 29''$, solve the trihedral. *Ans.* Impossible (135).

5. Given $\alpha = 153° \; 28' \; 33''$, $\beta = 82° \; 58' \; 17''$, $A = 29° \; 32' \; 29''$, solve the trihedral. *Ans.* Impossible (§ 138, VII, and (135)].

6. Given $\alpha = 97° \; 35'$, $\gamma = 27° \; 08' \; 22''$, $A = 40° \; 51' \; 18''$, solve the trihedral.

Ans. $C = 17° \; 31' \; 09''$, $B = 144° \; 48' \; 10''$, $\beta = 119° \; 08' \; 25''$.

7. Given $\alpha = 150° \; 57' \; 05''$, $\beta = 134° \; 15' \; 54''$, $A = 144° \; 22' \; 42''$.

find $B = 59° \; 12' \; 15''$, $C = 29° \; 8' \; 42''$,
$B'' = 120° \; 47' \; 45''$, $C'' = 97° \; 42' \; 48''$.

8. Given $\alpha = 99° \; 40' \; 48''$, $\beta = 64° \; 23' \; 15''$, $A = 95° \; 38' \; 4''$.

Ans. $\gamma = 100° \; 49' \; 30''$, $C = 97° \; 26' \; 29''$, $B = 65° \; 33' \; 10''$.

If B is near 90°, it cannot be accurately computed from its sine. The method by auxiliary right trihedrals (§ 129) may then be used.

143. CASE IV. *Given two dihedrals and a face-angle opposite one of them;* or A, B and α

The solution, when possible, may be made indirectly by passing to the polar trihedral which corresponds to Case III.: otherwise, thus—

To compute β. (135) gives

$$\sin \beta = \sin B \frac{\sin \alpha}{\sin A}.$$

Hence, $\log \sin \beta = \log \sin B + \log \sin \alpha - \log \sin A.$

1°. If $\log \sin \beta > 0$; $\sin \beta > 1$, which is impossible, and there is *no solution*.

2°. If $\log \sin \beta = 0$; $\sin \beta = 1$, $\beta = 90°$, giving but *one possible solution*, a quadrantal trihedral.

3°. If $\log \sin \beta < 0$; $\sin \beta < 1$, giving two supplementary angles, both of which are admissible only when B is nearer to 90° than A.

To compute C *and* γ. (160) and (162), Napier's Analogies, may be used as in Case III., and the results checked by (136).

Examples.

1. Given $A = 95° 16'$, $B = 80° 42' 10''$, $\alpha = 57° 38'$, solve the trihedral.

Ans. $\beta = 56° 49' 57''$, $\gamma = 6° 18' 18''$. $C = 7° 26' 21''$.

2. Given $A = 50° 12'$, $C = 58° 08'$, $\alpha = 62° 42'$, solve the trihedral.

Ans. $\gamma = 79° 12' 10''$, $\beta = 119° 03' 26''$, $B = 130° 54' 28''$;
$\gamma' = 100° 47' 50''$, $\beta' = 152° 14' 18''$, $B' = 156° 15' 06''$.

3. Given $B = 82° 58' 10''$, $C = 43° 27' 36''$, $\gamma = 29°, 32' 29''$, solve the trihedral.

Ans. $\beta = 45° 21' 05''$, $\alpha = 40° 16' 40''$, $A = 64° 24' 20''$;
$\beta' = 134° 38' 55''$, $\alpha' = 147° 29' 29''$, $A' = 131° 43' 40''$

4. Given $A = 61° 37' 53''$, $B = 139° 54' 34''.5$, $\beta = 150° 17' 26''.2$, solve the trihedral.

$$Ans.\quad \gamma = \begin{cases} 129° \; 40' \; 54'', \\ 19° \; 58' \; 44''; \end{cases}$$

$$\alpha = \begin{cases} 42° \; 37' \; 23'', \\ 137° \; 22' \; 37''; \end{cases}$$

$$C = \begin{cases} 89° \; 53' \; 36'', \\ 26° \; 21' \; 56''. \end{cases}$$

5. Given $A = 50° 12'$, $B = 58° 08'$, $\alpha = 162° 42'$.

Ans. Impossible.

If β is near 90°, it cannot be accurately computed from its sine. The computation by auxiliary right trihedrals (§ 129) should then be made.

144. CASE V. *Given two dihedrals and their included face-angle;* or A, B and γ.

To compute α and β, we have the third (161) and fourth (162) of Napier's Analogies, as follows:

$$\cos \tfrac{1}{2}(A + B) : \cos \tfrac{1}{2}(A - B) :: \tan \tfrac{1}{2}\gamma : \tan \tfrac{1}{2}(\alpha + \beta),$$

$$\sin \tfrac{1}{2}(A + B) : \sin \tfrac{1}{2}(A - B) :: \tan \tfrac{1}{2}\gamma : \tan \tfrac{1}{2}(\alpha - \beta);$$

from which $\frac{1}{2}(\alpha + \beta)$ and $\frac{1}{2}(\alpha - \beta)$ should be computed.

$$\alpha = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}(\alpha - \beta), \quad \beta = \tfrac{1}{2}(\alpha + \beta) - \tfrac{1}{2}(\alpha - \beta).$$

To compute C either the first (159) or second (160) of Napier's Analogies, or the proportion of the sines, may be used. The last should not be used if γ is nearer 90° than both α and β, as the species of C would not then be determined by principle VII of § 138. It is usually taken as a check formula.

Examples.

1. Given $A = 81° \, 38' \, 20''$, $B = 70° \, 09' \, 38''$, $\gamma = 59° \, 16' \, 22''$, solve the trihedral.

Ans. $\alpha = 70° \, 04' \, 17''$, $\beta .= 63° \, 21' \, 27''$, $C = 64° \, 46' \, 24''$

2. Given $A = 160° \, 20'$, $B = 33° \, 22' \, 10''$, $\gamma = 127° \, 33' \, 20''$, solve the trihedral.

Ans. $\alpha = 158° \, 50'' \, 22''$, $\beta = 36° \, 09' \, 18''$, $C = 47° \, 39' \, 22''$.

145. CASE VI. *Given the three dihedrals;* or A, B and C.

To compute α, β and γ, we may use (150), (151) or (152). If more than one face-angle is required, it is best to use (153) and (154).

$$H = \sqrt{\frac{-\cos \frac{1}{2}S}{\cos (\frac{1}{2}S - A) \cos (\frac{1}{2}S - B) \cos (\frac{1}{2} S - C)}};$$

$$\tan \tfrac{1}{2}\alpha = H \cos (\tfrac{1}{2}S - A).$$

It should be noted that by Principle IV. of § 138,

$$180° < S < 540°, \therefore 90° < \tfrac{1}{2}S < 270°,$$

which gives a positive value for $-\cos \frac{1}{2}S$; also by Principle V.,

$$A > B + C - 180°,$$

or $$180° > B + C - A, \therefore 90° > \tfrac{1}{2}(B + C - A),$$

or $90° > \frac{1}{2}S - A$, which gives a positive value for $\cos (\frac{1}{2}S - A)$. In like manner, each other factor of the value of H may be shown to be positive. H is therefore real for all values which satisfy the above tests.

The proportions of the sines (135) and (136) may be used as checks.

Examples.

1. Given $A = 65^\circ\ 28'\ 50''$, $B = 50^\circ\ 56'\ 40''$. $C = 73^\circ\ 37'\ 10''$.

$$\begin{array}{l} A = 65^\circ\ 28'\ 50'' \\ B = 50^\circ\ 56'\ 40'' \\ C = 73^\circ\ 37'\ 10'' \\ \hline S = 190^\circ\ 02'\ 40'' \end{array} \qquad H = \sqrt{\frac{-\cos\frac{1}{2}S}{\cos(\frac{1}{2}S - A)\cos(\frac{1}{2}S - B)\cos(\frac{1}{2}S - C)}}.$$

$$\begin{array}{ll} \frac{1}{2}S \qquad = 95^\circ\ 01'\ 20'' & + \log(-\cos\frac{1}{2}S) = 8.94222 \\ \frac{1}{2}S - A = 29^\circ\ 32'\ 30'' & \text{a. c. } \log\cos(\frac{1}{2}S - A) = 0\ 0604\hat{8} \\ \frac{1}{2}S - B = 44^\circ\ 04'\ 40'' & \text{a. c. } \log\cos(\frac{1}{2}S - B) = 0.1436\hat{3} \\ \frac{1}{2}S - C = 21^\circ\ 24'\ 10'' & \text{a. c. } \log\cos(\frac{1}{2}S - C) = \underline{0.0310\hat{3}} \\ & \qquad\qquad\qquad\qquad\quad \underline{9.1773\hat{7})2} \\ & \qquad\qquad\qquad\quad \log H = 9.58869 \end{array}$$

$$\begin{aligned} \tan\tfrac{1}{2}\alpha &= H\cos(\tfrac{1}{2}S - A), \\ \tan\tfrac{1}{2}\beta &= H\cos(\tfrac{1}{2}S - B), \\ \tan\tfrac{1}{2}\gamma &= H\cos(\tfrac{1}{2}S - C). \end{aligned}$$

$$\begin{array}{rrr} \log H = 9.58869 & 9.58869 & 9.58869 \\ \log\cos(\frac{1}{2}S - A) = \underline{9.9395\hat{1}} & \log\cos(\frac{1}{2}S - B) = \underline{9.8563\hat{6}} & \log\cos(\frac{1}{2}S - C) = \underline{9.9689\hat{6}} \\ 18^\circ\ 38'\ 48''\ \ 9.5282\hat{0} & 15^\circ\ 34'\ 13''\ \ 9.4450\hat{5} & 19^\circ\ 51'\ 21''\ \ 9.5576\hat{5} \end{array}$$

$$\alpha = 37^\circ\ 17'\ 36'', \quad \beta = 31^\circ\ 08'\ 26'', \quad \gamma = 39^\circ\ 42'\ 42''.$$

$$\frac{\sin\alpha}{\sin A} = \frac{\sin\beta}{\sin B} = \frac{\sin\gamma}{\sin C}.$$

$$\begin{array}{rrr} \log\sin\alpha = 9.78240 & \log\sin\beta = 9.7136\hat{0} & \log\sin\gamma = 9.80545 \\ \text{a. c. } \log\sin A = \underline{0.0410\hat{4}} & \text{a. c. } \log\sin B = \underline{0.10984} & \text{a. c. } \log\sin C = \underline{0.0179\hat{9}} \\ 9.8234\hat{4} & 9.8234\hat{4} & 9.8234\hat{4} \end{array}$$

2. Given $A = 48° \ 30'$, $B = 125° \ 20'$, $C = 62° \ 54'$, solve the trihedral.

Ans. $\alpha = 56° \ 39' \ 30''$, $\beta = 114° \ 29' \ 56''$, $\gamma = 83° \ 12' \ 4''$.

3. Given $A = 125° \ 20'$, $B = 162° \ 40'$, $C = 100° \ 10''$.

Ans. Impossible. Compare A with $180° - (B - C)$.

4. Given $A = 48° \ 30'$, $B = 125° \ 20'$, $C = 130° \ 54'$.

Ans. Impossible. Compare A with $(B + C) - 180°$.

146. *Solution of Oblique Spherical Triangles.* Any oblique spherical triangle should be solved by means of its corresponding oblique trihedral at the centre of the sphere. The radius of the sphere and any three given parts of the spherical triangle will determine the three corresponding parts of its trihedral (§ 110), which should be solved. The computed parts of the trihedral, together with the radius of the sphere, will determine the required parts of the spherical triangle.

Examples.

1. Given $a = 176.5449$, $b = 189.756$, $c = 261.9$, $R = 100$, solve the triangle.

36″ 12′ 3′ 58″

Ans. $\alpha = 101° \ 9' \ 10''$, $\beta = 108° \ 43' \ 30''$, $\gamma = 150° \ 5' \ 20''$;
$A = 177° \ 10' \ 58''$, $B = 177° \ 16' \ 50''$, $C = 178° \ 34' \ 06''$.

2. Given $a = 327.9473$, $b = 53.5234$, $C = 150° \ 24' \ 12''$, $R = 300$, solve the triangle.

Ans. $\alpha = 62° \ 38'$, $\beta = 10° \ 13' \ 20''$, $\gamma = 71° \ 37' \ 06''$;
$A = 27° \ 31' \ 46''$, $B = 5° \ 17' \ 58''$, $c = 374.9199$.

3. Given $a = 34.5587$, $b = 51.948$, $A = 58° \ 08'$, $R = 25$, solve the triangle.

Ans. $\alpha = 79° \ 12' \ 10''$, $\beta = 119° \ 03' \ 26''$, $\gamma = 62° \ 41' \ 58''$;
$B = 130° \ 54' \ 30''$, $C = 50° \ 11' \ 56''$, $c = 27.3585$.

4. Given $a = 109.4321$, $A = 50° \ 12'$, $C = 58° \ 08'$, $R = 100$, solve the triangle.

Ans. $\alpha = 62° \ 42'$, $\beta = 119° \ 03' \ 26''$, $\gamma = 79° \ 12' \ 10''$;
$\beta' = 152° \ 14' \ 18''$, $\gamma' = 100° \ 47' \ 50''$;
$B = 130° \ 54' \ 28''$, $b = 208.6919$, $c = 138.2349$;
$B' = 156° \ 15' \ 06''$, $b' = 265.3317$, $c' = 175.9244$.

5. Given $c = 413.802$, $A = 81° 38' 20''$, $B = 70° 09' 38''$, $R = 400$, solve the triangle.

Ans. $\alpha = 70° 04' 17''$, $\beta = 63° 21' 27''$, $\gamma = 59° 16' 22''$;
$a = 489.1905$, $b = 442.3188$, $C = 64° 46' 24''$.

6. Given $A = 65° 28' 50''$, $B = 50° 56' 40''$, $C = 73° 37' 10''$, $R = 500$, solve the triangle.

Ans. $a = 325.446$, $b = 271.753$, $c = 346.55$.

147. *Area of a Spherical Triangle from its Angles.* Denote by Δ the area of any spherical triangle, and by E its spherical excess expressed in radians.

From Geometry, the area of the triangle is equal to that of a tri-rectangular triangle [i.e., $\frac{1}{8}$ the surface of the sphere] multiplied by the spherical excess expressed in right angles, i.e., by $E \times \frac{2}{\pi}$.

Hence,
$$\Delta = \frac{4\pi R^2}{8} \times \frac{2}{\pi} E,$$

or
$$\Delta = ER^2.$$

That is: *the area of any spherical triangle is equal to its spherical excess in radians multiplied by the square of the radius of the sphere.*

The angles of the triangle are usually given or computed in degrees, and

$$A + B + C - 180°$$

is the spherical excess in degrees. Hence,

$$E = \frac{\pi}{180}[A + B + C - 180°]$$

gives the excess in radians.

The area may be found when any three parts of the triangle are given, by computing such angles as are not given.

Examples.

1. Given $A = 95^\circ\ 16'$, $B = 80^\circ\ 42'\ 10''$, $C = 7^\circ\ 26'\ 21''$, $R = 10$ ft., find the area of the spherical triangle. *Ans.* 5.945 sq. ft

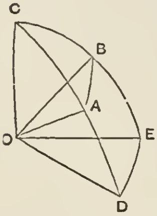
2. Given $a = 109.3158$, $b = 17.8411$, $C = 150^\circ\ 24'\ 12''$, $R = 100$ yds., find the area of the spherical triangle. *Ans.* 564.128 sq. yds.

3. Given $R = 150$ ft., $a = 61$ ft., $b = 43$ ft., and $c = 21$ ft., compute the area of the spherical triangle.

Ans. $A = 142^\circ\ 38'\ 22''$, $B = 25^\circ\ 42'\ 32''$, $C = 12^\circ\ 21'\ 40''$,
$\triangle = 278.6$ sq. ft.

Applications.

148. I. *To find the angle between two stars when it cannot be measured directly.*



Let O be the point of observation, OA and OB the lines of sight to the two stars. Let vertical planes through OA and OB intersect in OC, and cut OD and OE from the horizontal plane through O.

Measure directly the horizontal angle DOE between the vertical planes, also the angles DOA and EOB of the stars above the horizontal plane ODE.

OA, OB and OC are the edges of a trihedral in which $\alpha = 90^\circ - EOB$, $\beta = 90^\circ - DOA$, $C = DOE$, all being given by measurement. The required angle γ may then be computed (§ 141).

Examples.

1. Given (fig.) $DOA = 79^\circ\ 46'\ 40''$, $EOB = 27^\circ\ 22'$, $DOE = 150^\circ\ 24'\ 12''$, compute AOB. *Ans.* $71^\circ\ 37'\ 04''$.

2. Given (fig.) $DOA = 52^\circ\ 50'$, $EOB = 21^\circ\ 13'\ 58''$, $DOE = 39^\circ\ 23'$, compute AOB. *Ans.* $43^\circ\ 37'\ 38''$.

II. *Knowing the radius of the earth, and the latitudes and longitudes of two points on its surface, to find the length of the arc of a great circle joining them.*

Let A and B be the two points, P the north pole. Through P and A, P and B, A and B draw arcs of great circles, those through P cutting the equator $E'CE$ at C and D respectively.

Let ϕ = latitude of A,
λ = longitude of A,
ϕ' = latitude of B,
λ' = longitude of B;

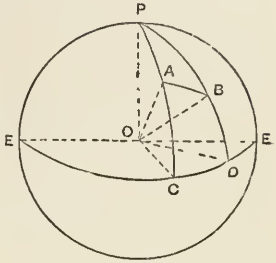
all given in degrees, minutes and seconds.

Suppose PE to be the meridian from which the longitude is estimated. Then, from the figure, we have

$$\lambda = EOC, \quad \phi = COA;$$
$$\lambda' = EOD, \quad \phi' = DOB.$$

In the trihedral corresponding to ABP—

The dihedral at P $= \lambda - \lambda'$.
The face-angle $AOP = 90° - \phi$.
The face-angle $BOP = 90° - \phi'$.



From these (§ 141) compute the face-angle AOB in degrees.

Let R = 3963 English [or statute] miles.

To find AB in miles, it is only necessary to reduce AOB to radians, and multiply it by R.

$$AB = AOB\,\frac{\pi}{180}\,R = AOB\,\frac{3.1416}{180}\,3963 = AOB \times 69.167 +,$$

or

$$AB = 69\tfrac{1}{6} \times AOB \text{ in degrees.}$$

In the following examples the above value of the radius will be considered as given, and this formula may be used.

Examples.

1. Given, latitude of New York City	40° 43′ 48″.5 N.
longitude of “ “	73° 59′ 02″.6 W.
latitude of Liverpool	53° 24′ 47″.7 N.
longitude of “	3° 0′ 0″ W.

Find the length of the arc of a great circle joining them [in statute miles]. *Ans.* 3309.5 statute miles.

2. Given, latitude of Sandy Hook 40° 30′ N.
longitude of " " 74° 5′ W.

latitude of Queenstown 51° 50′ N.
longitude of " 8° 10′ W.

Find the length [in statute miles] of the arc of a great circle joining them. *Ans.* 3139.05 statute miles.

3. Given, latitude of Berlin 52° 31′ 13″ N.
longitude of " 13° 23′ 52″ E.

latitude of Washington 38° 53′ 38″.8 N.
longitude of " 77° 3′ 1″.8 W.

Find the shortest distance [in statute miles] on the surface of the earth from Washington to Berlin. *Ans.* 4174.8 stat. mi.

4. If a ship sails from Boston, lat. 42° 22′ N., long. 71° 16′ W., starting due East, and continues her course on the arc of a great circle, what will be her latitude when she reaches the meridian of Greenwich, and in what direction will she then be sailing?

Ans. { Lat. 16° 19′ 35″ N.
S. of E. 129° 39′ 20″.

5. A ship sails from Valparaiso, Chili, lat. 33° 02′ S., long. 71° 43′ W., starting in a northwesterly direction, and continues on the arc of a great circle. When she has travelled 3480 [statute] miles, what are her latitude and longitude?

Lat. 3° 39′ 40″ N.
Long. 108° 6′ 59″ W.

6. A ship sails from Rio Janeiro, Brazil, lat. 23° S., long. 43° W., and runs 5624.4 statute miles in a general northeasterly direction on the arc of a great circle, when her latitude is found to be 50° N. What is her longitude? *Ans.* 2° 33′ 30″ W.

7. A ship sails from San Francisco, Cal., lat. 37° 40′ N., long. 122° 20′ W., in a general southwestward direction, on the arc of a great circle. When she has gone 2460 miles, her longitude is 160° W.; what is her latitude? *Ans.* 22° 37′ 20″ N.

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